Chapter 8

Mean Field Theory

The concept of mean field theory is widely used for the description of interacting many-body systems in physics. The idea behind is to treat the many-body system not by summing up all mutual two-body interactions of the particles but to describe the interaction of one particle with the remaining ones by an average potential created by the other particles. Let’s consider particle $i$ at position $\vec{r}_i$ which feels the potential $U$ created by particles $j$

$$U(\vec{r}_i) = \sum_j V(\vec{r}_i, \vec{r}_j) \rightarrow U[\rho(\vec{r}_i)] .$$

Then particle $i$ feels an average potential which depends on the particle density

$$\rho(\vec{r}) = \sum_j \Psi_j^*(\vec{r})\Psi_j(\vec{r})$$

at its position. The problem is now treated in terms of a mean field potential $U[\rho]$ which is a functional of the density $\rho$. Mean field theory is very efficient for the description of many-body systems like finite nuclei or infinite nuclear matter, respectively neutron matter which occurs in the interior of neutron stars. The task is now to find the correct density functional $U[\rho]$ which minimises the many-body Hamiltonian. That such a density functional exists can be proven within density functional theory (Kohn-Sham Theory [2, 1]), how it looks like depends on the problem. To find a functional which comes as close as possible to the - in principle existing - exact solution is the task to be solved what is in general very difficult. An additional caveat for nuclear systems is thereby that the existence theorem, the Hohenberg-Kohn theorem [3] has been proven for particles in an external field, e.g. atoms in the electromagnetic field. That this theorem holds as well for self-bound systems, such as nuclei, is assumed but has not been proven explicitly.

In the following we discuss well established and very successful relativistic mean field models for nuclear systems. However, before coming to the models in detail, we briefly sketch some basic features of nuclear many-body systems.
8.1 Basic features of infinite nuclear matter

The simplest, however, already highly non-trivial system, is infinite nuclear matter or infinite neutron matter. In this case the infinite system is homogeneous and isotropic and therefore the wave functions are given by plane waves $e^{i \mathbf{k} \cdot \mathbf{x}}$. The particle density is obtained by the sum over all occupied states inside the phase space volume $(2\pi)^3$

$$\rho = \frac{N}{V} = \sum_{k,\lambda} \Psi_{k,\lambda}^*(\mathbf{r}) \Psi_{k,\lambda}(\mathbf{r}) \mapsto \frac{\gamma}{(2\pi)^3} \int d^3k \ n(\mathbf{k}) \ .$$ (8.1)

In the continuum limit the sum in (8.1) is replaced by the integral over the momentum distribution $n(\mathbf{k})$. Since we deal with fermions the quantum states inside the volume $(2\pi)^3$ have to be different. Therefore all states $\mathbf{k}$ are occupied up to the Fermi momentum $k_F$. Therefore the distribution of occupied states if given by the Fermi sphere with radius $k_F$

$$n(\mathbf{k}) = \Theta(k_F - |\mathbf{k}|) \ .$$

Evaluating (8.1) leads to the following relation between density and Fermi momentum

$$\rho = \frac{\gamma k_F^3}{6\pi^2} \quad \text{(8.2)}$$

where $\gamma$ is a degeneracy factor. For isospin symmetric nuclear matter with equal number of protons and neutrons is $\gamma = 4$ which means that each momentum state $\mathbf{k}$ can be occupied
8.1. BASIC FEATURES OF INFINITE NUCLEAR MATTER

by four states (proton and neutron, both with spin up/down). Consequently, for pure neutron matter the spin-isospin degeneracy is $\gamma = 2$.

The key quantity which describes the properties of infinite matter is the Equation of State (EOS), i.e. the energy density $\epsilon$ as a function of particle density. The energy density is just the sum of kinetic energy and the mean field

$$\epsilon(\rho) = \frac{\gamma}{(2\pi)^3} \int d^3k \frac{k^2}{2M} n(k) + U[\rho]$$

One can also characterise the equation of state in terms of pressure density instead of energy density which is usually done in hydrodynamics when ideal fluids or gases are described. Both descriptions are equivalent since energy density and pressure are related by thermodynamical relations. In nuclear physics, however, it is more practical to use the energy density or even the energy per particle

$$E/A = \epsilon/\rho = \frac{3k_F^2}{10M} + U[\rho]/\rho$$

(8.3)

to characterise nuclear matter. In (8.3) we evaluated the integral for the kinetic energy and it should be noted that this is the non-relativistic expression for the energy per particle. From the existence of stable nuclei it follows that the energy per particle $E/A$ must have a minimum. This point is called the saturation point. The value, i.e. the binding energy per nucleon, can be derived from the Weizäcker mass formula for finite nuclei

$$E = -a_1A + a_2A^{2/3} + a_3\frac{Z^2}{A^{1/3}} + a_4\frac{(A - Z)^2}{A} + \lambda\frac{a_5}{A^{3/4}}$$

(8.4)

Just as a reminder, the Weizäcker mass formula contains five terms and with these five terms it provides a rather accurate fit to the periodic table of stable nuclei. The contributions are: the volume term describing the nuclear bulk properties, i.e. the conditions...
in the interior of a heavy nucleus. The surface term accounts for the surface tension as it exists also in a liquid drop. The symmetry term arises from the isospin dependence of the nuclear forces (exchange of isovector mesons) and it scales with the difference of proton and neutron number. The pairing term is due to the phenomenon of super-fluidity which exists also in nuclear systems and will be discussed in detail in Chapter XXX.

Since we consider infinite, isotropic and homogeneous nuclear matter only the volume term contributes and the Weizäcker mass formula gives us a value of $E/A \simeq -16$ MeV at the saturation point. Also the density where this minimum occurs is known from electron scattering on finite nuclei. It is the density in the interior of heavy nuclei, e.g. in $^{208}$Pb. This so-called saturation density is about $\rho_0 \simeq 0.16$ fm$^{-3}$.

Thus a key requirement for a realistic density functional is to meet the nuclear saturation properties:

\[
\rho_0 \simeq 0.16 \text{ fm}^{-3}; \quad k_{F_0} \simeq 1.36 \text{ fm}^{-1} = 260 \text{ MeV}; \quad \text{and} \quad E/A(\rho_0) \simeq -16 \text{ MeV}.
\]

Figure 8.3: Nuclear matter at supra-normal densities does not exists in nature except in neutron stars and supernovae explosions (left: Chandra X-Ray space telescope picture of the Crab nebula with a radio pulsar neutron star in the centre, the remnant of a supernova explosion of a star with 8 – 12 solar masses in the year 1054.). For short times superdense matter can be created on earth by means of relativistic heavy ion collisions (right: simulation of a relativistic heavy ion collision of Au+Au at the Relativistic Heavy Ion Collider RHIC at Brookhaven/USA).

Intuitively the value of the saturation density is easy to understand: with a radius of about 1.2 fm the volume occupied by a nucleon is about 8 fm$^3$ and hence the density where the nucleons start to touch is $1/8$ fm$^{-3}$. From the Van der Waal’s like behaviour of the nuclear forces (see Fig. 7.3) it is intuitively clear that the configuration where the nucleons
start to touch is the energetically most favourable one. Here the contribution from the strong intermediate range attraction is maximal. If the matter is further compressed the nucleons start to feel the repulsive hard core of the potential. This is the reason why under normal conditions nuclear systems do not exist in nature at densities which exceed the saturation density $\rho_0$. However, there exists one exception and these are neutron stars. In neutron stars the gravitational pressure is able to compress nuclear matter up to five to ten times saturation density as model calculations show. In such model calculations the neutron star properties depend crucially on the high density behaviour of the nuclear EOS. The investigation of such highly compressed dense matter is therefore a hot topic of present research, both theoretically and experimentally. From the experimental side the only way to create supra-normal densities are energetic heavy ion collisions where two ions are accelerated and shot on top of each other. One believes that in such reactions densities between two up to ten times saturation density are reached, depending on the bombarding energy. The problem is, however, that the dense system exists only for a very short period before it explodes.
8.2 The $\sigma\omega$-model

The $\sigma\omega$-model is a very transparent and efficient model, describing nuclear matter, neutron stars and finite nuclei. The $\sigma\omega$-model contains only two mesons, namely the well known scalar $\sigma$ and vector $\omega$ meson. Based on only these two mesons it is the simplest version of Quantum Hadron Dynamics (QHD) which contains, however, already at that level all relevant aspects of relativistic nuclear dynamics. Sometimes the $\sigma\omega$-model is also called Walecka model, who developed the first version of QHD in 1974 [4]. However, the original idea of an effective scalar and vector exchange goes even back to the year of 1956 (Dürr 1956 [5]). Originally the attempt of QHD was to formulate a renormalizable meson theory of strong interactions. In the meantime it is, however, considered as an effective theory which should only be applied at the mean field level. It is effective in the sense that the coupling constants of the $\sigma$ and $\omega$ mesons with the nucleon are not determined from free nucleon-nucleon scattering as in the case of the boson-exchange potential discussed in the previous chapter but are treated as free parameters which can be adjusted to the properties of nuclear matter, in particular the nuclear saturation point. In this sense the $\sigma\omega$-model provides the simplest form of a relativistic density functional for the nuclear EOS (8.3).

First we give some literature concerning the $\sigma\omega$-model:

- J.D. Walecka, ”Theoretical Nuclear and Subnuclear Physics”, Oxford 1995
- B.D. Serot & J.D. Walecka, ”Advances in Nuclear Phys.”, Calderon Press 1986

We start with a short overview of the degrees of freedom:

- nucleon
- scalar $\sigma$-meson (iso-scalar)
- vector $\omega$-meson (iso-scalar)
- (vector $\rho$-meson) (iso-vector)

In the following we neglect the $\rho$-meson which is of importance for a very accurate description of finite nuclei properties (single particle spectra, neutron skins etc.) but does not contribute in infinite nuclear matter.

Looking at the list of degrees of freedom, one might now ask: where is the pion? In the previous section we learned that the pion plays a very important role for the interactions in NN scattering. The reason is here that the pion exchange does not contribute to the potential at the mean field level but only by exchange terms (Fock-diagrams). The philosophy of an effective model like the $\sigma\omega$-model is to treat contributions which are beyond the approximation scheme of the model not explicitly but to absorb them in some way into the model parameters, i.e. into the coupling constants. Therefore the pion is not included as an explicit degree of freedom.
8.2. THE $\sigma\omega$-MODEL

8.2.1 Lagrange density and field equations

Before deriving the field equations for nucleons and mesons, the Lagrangian density has to be introduced

\[
\mathcal{L} = - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_{\mu} \omega^\mu + \frac{1}{2} \left\{ (\partial_\mu \sigma)^2 - m_\sigma^2 \sigma^2 \right\} + \bar{\psi} \gamma_\mu i \partial^\mu \psi + M \bar{\psi} \psi
\]

\[ - g_\omega \bar{\psi} \gamma_\mu \psi \omega^\mu + g_\sigma \bar{\psi} \psi \sigma \]  

(8.5)

Next, we will identify the different terms in (8.5). The first line of Eq. (8.5) contains the Lagrangian of free nucleons and mesons: the first and the third term represent the kinetic energy of mesons, whereas the second and 4th term stand for the meson rest energy. Correspondingly, the other two terms are the nucleon kinetic energy $\bar{\psi} \partial_\mu \gamma^\mu \psi$ and rest energy $M \bar{\psi} \psi$. The second line contains the interaction part $\bar{\psi} (-\gamma_\mu g_\omega \omega^\mu + g_\sigma \sigma) \psi$.

We are familiar with the field-strength-tensor, defined by

\[
F_{\mu\nu} = \partial_\mu \omega^\nu - \partial_\nu \omega^\mu 
\]

(8.6)

and we saw earlier that $F^{\mu\nu}$ is antisymmetric

\[
F^{\mu\nu} = -F^{\nu\mu} .
\]

The field equations result from the Euler-Lagrange equations

\[
\frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial (\partial \Phi / \partial x_\mu)} - \frac{\partial \mathcal{L}}{\partial \Phi}
\]

where $\Phi$ can be substituted by $\psi, \bar{\psi}, \sigma, \omega^\mu$.

We obtain the Dirac equation in the medium:

\[
[\gamma_\mu (i \partial^\mu - g_\omega \omega^\mu) - (M - g_\sigma \sigma)] \psi = 0 .
\]

(8.7)

The meson-fields enter into the Dirac equation as one would expect from minimal substitution known from electrodynamics. The scalar field which has no analog in electrodynamics goes into the mass:

scalar-field $\Rightarrow$ mass

vector-field $\Rightarrow$ appears in the derivative $\partial_\mu \rightarrow D_\mu = \partial_\mu + ig_\omega \omega_\mu$

Now, the meson-field equations read:

1. The Klein-Gordon equation with a source-term:

\[
[\partial_\mu \partial^\mu + m_\sigma^2] \sigma = g_\sigma \bar{\psi} \psi
\]

(8.8)

2. The Proca equation with a source-term $\equiv$ massive Maxwell equation:

\[
\partial^\nu F_{\mu\nu} + m_\omega^2 \omega_\mu = g_\omega \bar{\psi} \gamma_\mu \psi
\]

(8.9)
Both source terms are proportional to the corresponding densities where one has to distinguish between the scalar density
\[ \hat{\rho}_S = \bar{\psi}\psi \]
and the four-vector baryon current
\[ \hat{j}_\mu = \bar{\psi}\gamma_\mu\psi = (\hat{\rho}_B, \hat{\vec{j}}) .\]
Here, \( \hat{\rho}_B \) is the baryon density, whereas \( \hat{\vec{j}} \) represents the vector current.

The distinction between the scalar density \( \hat{\rho}_S \) and the time-like baryon density \( \hat{\rho}_B \) is a novel and essential feature of a relativistic description. This has severe consequences for the entire dynamics, as we will see later on.

The energy-momentum-tensor is given by
\[ T^{\mu\nu} = \mathcal{L}\delta^{\mu\nu} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi . \]
For isotropic and homogeneous systems, \( T^{\mu\nu} \) takes the form
\[ T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} \]
with the four-velocity \( u_\mu = (\gamma, \gamma\vec{v}) \), which becomes \((1, \vec{0})\) in the local rest frame of the matter. \( \epsilon \) stands for the energy density and \( P \) for the pressure.

8.2.2 Mean field theory
To introduce the Mean-Field Theory, we consider a volume \( V \), filled with \( N \) nucleons. Consequently, we have a baryon density of \( \rho_B = \frac{N}{V} \). If \( V \to 0 \), the baryon density increases correspondingly when \( N \) is kept constant. For this reason, the source terms in (8.8) and (8.9) become huge compared to quantum fluctuations.

As a consequence, the meson fields and the corresponding source terms can be substituted by their classical expectation values. More precisely, only the nucleon fields \( \bar{\psi} \) and \( \psi \) are quantised while the meson fields are treated as classical fields.

\[ \hat{\sigma} \to \langle\hat{\sigma}\rangle \equiv \psi \]
\[ \hat{\omega}_\mu \to \langle\hat{\omega}_\mu\rangle \equiv V_\mu \]

Considering nuclear matter in the rest frame (the natural one), the spacial components of the baryon current vanish
\[ \langle\hat{j}_\mu\rangle = j_\mu^\text{rf} = (\hat{\rho}_B, \vec{0}) . \]

Now we can derive the mean field potentials. Using (8.9) and (8.8) respectively, we get
\[ V_\mu = V_0 \delta_{\mu 0} = \frac{g_\omega}{m_\omega^2} \rho_B \]

and

\[ \Phi = \frac{g_\sigma}{m_\sigma^2} \rho_S . \]

Thus, in momentum space the Dirac equation reads in mean-field approximation

\[
\left( \vec{\alpha} \cdot \vec{k} + \beta M^* \right) u(\vec{k}) = (E - g_\omega V_0) u(\vec{k}) \]

(8.10)

where the energy eigenvalues are given by

\[
E = \pm \sqrt{\vec{k}^2 + M^*^2 + g_\omega V_0} \]

(8.11)

(\pm stands again for particles and anti-particles). The effective mass is

\[
M^* = M - g_\sigma \Phi . \]

(8.12)

The baryon field is equal to its expression in vacuum

\[
\hat{\psi}(\vec{x}) = \frac{1}{\sqrt{V}} \sum_{\vec{k} \lambda} \left[ u_\lambda(\vec{k}) \hat{a}_{\vec{k} \lambda} e^{i\vec{k}\vec{x}} + v_\lambda(-\vec{k}) \hat{b}^\dagger_{\vec{k} \lambda} e^{-i\vec{k}\vec{x}} \right]
\]

where \( \hat{a} \) and \( \hat{b} \) are particle and anti-particle annihilation operators, \( \hat{a}^\dagger \) and \( \hat{b}^\dagger \) the corresponding creation operators. However, now the nucleons are so-called \textit{quasi-particles} which are dressed by the interaction with the medium. This is reflected by the fact that the nucleons carry an effective mass (8.12)

\[
\bar{u}(\vec{k})u(\vec{k}) = \frac{M^*}{\sqrt{\vec{k}^2 + M^*^2}} u^\dagger(\vec{k}) u(\vec{k}) \]

but the structure of the nucleon spinors inside the medium is identical to those in the vacuum with the replacement \( M \rightarrow M^* \). If we further introduce an effective energy \( E^* = \sqrt{\vec{k}^2 + M^*^2} \) we can write the equations in the medium in a very transparent way. (For the sake of covariance we distinguish for the moment between \( \vec{k} \) and \( \vec{k}^* \) although the spatial components of the vector field vanish in mean field theory of nuclear matter, i.e. \( \vec{V} = 0 \) and both momenta are actually equal \( \vec{k} = \vec{k}^* \).) The energy eigenvalues (8.11) are given by \( E = \pm E^* + g_\omega V_0 \). Multiplying the Dirac equation (8.28) from the left with \( \gamma_0 \) we can rewrite it in the covariant form

\[
(\gamma_\mu k^{*\mu} - M^*) u(\vec{k}) = 0 \]

(8.13)

with the effective four-momentum

\[
k^{*\mu} = k^{\mu} - g_\omega V^{\mu} = (E^*, \vec{k}^*) . \]
The comparison with (8.7) shows that the effective derivative \( D_\mu \) transforms in momentum space to the effective momentum \( k^*_\mu \)

\[ iD_\mu = i\partial_\mu - g_\omega \omega_\mu \rightarrow k^*_\mu = k_\mu - g_\omega V_\mu . \]

From these considerations follows that the particles obey now a new mass-shell condition, namely

\[ k^2 - M^2 = 0 \rightarrow k^{*2} - M^{*2} = 0 . \]

The difference between the in-medium mass-shell condition and that in free space can be expressed in terms of the optical potential. After some simple algebra one finds

\[ 0 = k^{*2} - M^{*2} = k^2 - M^2 - 2MU_{\text{opt}} , \]

with

\[ U_{\text{opt}}(\rho, \vec{k}) = -g_\sigma \Phi + \frac{k_\mu g_\omega V^\mu}{M} + \frac{(g_\sigma \Phi)^2 - (g_\omega V_\mu)^2}{2M} \] (8.14)

The optical potential \( U_{\text{opt}} \) given by (8.14) is covariant and a Lorentz scalar. It is also called the Schrödinger-equivalent optical potential since it is exactly that potential which occurs when the non-relativistic Schrödinger equation is derived from the in-medium Dirac equation (8.7). If we insert the single particle energy \( k_0 = E \) from (8.11) into (8.14) we obtain the optical potential to leading order in the fields (where quadratic terms and terms which go with \( 1/M \) have been neglected):

\[ U_{\text{opt}} \simeq -g_\sigma \Phi + g_\omega V_0 . \] (8.15)

**Remark: baryon-density**

Computing the baryon-density one has to take into account the contribution from anti-particles. As we learned in XXX in the case of the electrons, in the vacuum all anti-particle states are occupied (filled Dirac sea). Using Wick’s theorem (see Chapter XXX) these contributions can be separated

\[ \hat{\rho}_B = \frac{1}{V} \sum_{\vec{k}, \lambda} \left( a_{\vec{k}, \lambda} a_{\vec{k}, \lambda}^\dagger - b_{\vec{k}, \lambda} b_{\vec{k}, \lambda}^\dagger \right) \]

=: \( \hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x}) \) normal-ordered product

\[ = \hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x}) - \left\langle 0 | \hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x}) | 0 \right\rangle \]

vacuum state = filled Dirac-sea
8.3 Nuclear and Neutron Matter

8.3.1 The Equation of State

In this subsection we deal with infinite nuclear matter and neutron matter. Nuclear matter consists of an equal number of protons and neutrons, i.e. it is an isospin symmetric system. All states are filled up to the Fermi momentum \( k_F \). The Fermi energy is the energy of a nucleon sitting at the Fermi surface, i.e. that of a nucleon with momentum \( |\vec{k}| = k_F \) and it is given by

\[
E_F = E_F^* + g\omega V_0 = \sqrt{k_F^2 + M^*^2} + g\omega V_0 .
\]  
(8.16)

Both, protons and neutrons have spin-up and spin-down states. Consequently, each proton-neutron-state is four times degenerate. Therefore, we have \( \gamma = 4 \) in the following equations, describing the energy density \( \epsilon \) and \( \rho_B \).

\[
\epsilon(k_F) = \frac{g_\omega^2}{2m_\omega} \rho_B^2 + \frac{m_\sigma^2}{2g_\sigma^2} (M - M^*)^2 + \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3 k \sqrt{k^2 + M^*^2} \]  
(8.17)

\[
\rho_B = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3 k = \frac{\gamma}{6\pi^2} k_F^3 .
\]

The equations for the scalar density and the effective mass are given by

\[
\rho_S = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3 k \frac{M^*}{\sqrt{k^2 + M^*^2}}
\]

\[
= \frac{\gamma}{4\pi} M^* \left[ k_F \sqrt{k_F^2 + M^*^2} - M^* \ln \frac{k_F + \sqrt{k_F^2 + M^*^2}}{M^*} \right] \]  
(8.18)

\[
M^* = M - g_\sigma \Phi = M - \frac{g_\omega^2}{m_\omega^2} \rho_S .
\]  
(8.19)

These two equations cannot be decoupled. In such a case one speaks about self-consistent equations which have to be solved iteratively. This means that one has to solve this set of coupled equations numerically.

Remark: self-consistent equations

Self-consistent equations are typical for problems which cannot be solved within perturbation theory. The solution techniques are, however, not difficult. One chooses a starting value for \( M^* \), usually the free mass, which is inserted into the equation for \( \rho_S \). The
value for \( \rho_S \) is then used to calculate the new \( M^* \) and this value is reinserted into the equation for \( \rho_S \) and so on. This procedure is repeated until convergence is received.

Figure 8.4: Equation of state (left) and effective mass (right) for nuclear and neutron matter in the \( \sigma\omega \)-model (QHD-I). The yellow area in the left panel indicates the empirical region of saturation.

To continue we evaluate as a next step the integral for the kinetic energy:

\[
\epsilon_{\text{kin}} = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3 k \sqrt{k^2 + M^{*2}} = \frac{\gamma}{(2\pi)^3} 4\pi \int_0^{k_F} dk \frac{k^2}{\sqrt{k^2 + M^{*2}}}
\]

\[
= \alpha \int dk \frac{k^2(k^2 + M^{*2})}{\sqrt{k^2 + M^{*2}}} = \alpha \int d^3 k \frac{k^2}{E^*} - M^* \rho_S
\]

\[
= M^* \rho_S + \alpha k^3 E^* \bigg|_0^{k_F} - 3\alpha \int_0^{k_F} dk^2 E^*
\]
and obtain
\[
\epsilon_{\text{kin}} = \frac{3}{4} \rho_B E^*_{F}(k_F) + \frac{1}{4} M^* \rho_S(k_F).
\] (8.20)

The solution of (8.19) leads to the equation-of-state (EOS). Now we can write down the energy density of nuclear matter
\[
\epsilon(k_F) = \frac{g_\omega}{2} V_0 \rho_B + \frac{g_\sigma}{2} \Phi \rho_S + \frac{3}{4} E^*_F \rho_B + \frac{1}{4} M^* \rho_S.
\] (8.21)

where the effective Fermi energy is given by \( E^*_F = \sqrt{k^2_F + M^*} \). Like in the non-relativistic case, the binding energy \( E_B \) or the energy per particle is obtained by
\[
E_B(\rho_B) = \frac{E}{A} - M = \frac{\epsilon}{\rho_B} - M.
\]

The only difference is that we have to subtract the rest mass of the nucleons which is still included in the kinetic energy. The resulting EOS for nuclear and neutron matter are shown in Figure 8.4. In the case of nuclear matter also the empirical region of saturation is depicted. We see that the minimum of the EOS is close to that region but shifted a little bit to too large densities. This is a consequence of the minimal form of QHD with only two free parameters as represented by the \( \sigma\omega \)-model discussed here (QHD-I). There exist many extensions of the \( \sigma\omega \)-model where this behaviour is then significantly improved. In contrast to nuclear matter, neutron matter is always unbound. The reason is that at equal density the by a factor two smaller degeneracy leads to a Fermi momentum for neutron matter which is larger by a factor \( 2^{1/3} \). Consequently the average kinetic energy of the neutrons is too large to bind this system. Hence the EOS reflects the fact that no nuclei exist in nature which consist only of neutrons. This is a direct consequence of the Pauli principle of quantum mechanics.

In addition the effective masses are shown in this figure. We see that the effective masses are dramatically reduced at finite densities but do only weakly depend on the isospin degree of freedom.

A quantity which is used to characterise the stiffness of the EOS, i.e. how strongly it increases with density, is the so-called incompressibility or compression modulus. The compression modulus \( K \) is a measure for the energy needed to compress the matter. It follows from a Taylor expansion of the energy per particle around the saturation point
\[
E_B = a_4 + \frac{L}{3} \left( \frac{\rho_B - \rho_0}{\rho_0} \right) + \frac{K}{18} \left( \frac{\rho_B - \rho_0}{\rho_0} \right)^2 + \cdots.
\]

The slope parameter \( L \) follows then as the first derivative of \( E_B \)
\[
L = 3 \rho_0 \frac{\partial E_B}{\partial \rho_B} \bigg|_{\rho_B=\rho_0} = \frac{3}{\rho_0} P(\rho_0).
\]
Per definition the pressure vanishes at the saturation point \((P(\rho_0) = 0)\) and thus at this point \(E_B\) is determined by the curvature, i.e. the slope of the pressure which is just the compression modulus \(K\):

\[
K = 9\frac{\partial P}{\partial \rho_B}\bigg|_{\rho_B=\rho_0} = 9\rho_B^2 \frac{\partial^2 (e/\rho_B)}{\partial \rho_B^2}\bigg|_{\rho_B=\rho_0} .
\]  

(8.22)

Taking the second derivative of (8.21) one obtains

\[
K = 9\rho_0 \left[ \frac{k_F^2}{3E_F^* \rho_B} + \frac{g_2^2}{m^2_\omega} - \frac{g_2^2 M^*^2}{m^2_\omega E_F^*^2} \right]_{\rho_B=\rho_0} .
\]

(8.23)

The first term arises from the kinetic pressure of a Fermi gas, the remaining terms are due to the potential. In QHD-I one obtains a value of \(K = 540\) MeV at \(\rho_0\) which is rather large. This means that the corresponding EOS of symmetric matter is very stiff, i.e. very repulsive at higher densities. Empirically the compression modulus can be deduced from vibration modes of excited nuclei, so-called giant resonances. In giant resonances the nucleus performs density fluctuations around its ground state. The corresponding value of \(K\) is about \(210 \pm 20\) MeV.

### 8.3.2 Expansion into the Fermi momentum \(k_F\)

For the following discussion it is instructive to perform an expansion of the energy density, Eq. (8.21) in terms of the Fermi momentum. Such an expansion is typically a low density expansion which makes sense as long as the criterium \(k_F << M^*\) is fulfilled. First we evaluate the scalar density:

\[
\rho_S = \gamma \frac{(2\pi)^3}{3} \int_0^{k_F} d^3k \frac{M^*}{\sqrt{k^2 + M^*^2}}
\]

\[
= \gamma \frac{(2\pi)^3}{3} \int_0^{k_F} d^3k \frac{1}{\sqrt{1 + \left( \frac{k}{M^*} \right)^2}} \approx \gamma \frac{(2\pi)^3}{3} \int_0^{k_F} d^3k \left[ 1 - \frac{k^2}{2M^*^2} \right]
\]

inserting

\[
\frac{k^2}{2M^*} = \frac{k^2}{2 \left( M - \frac{g_2^2}{m^2_\omega} \rho_S \right)^2} \approx \frac{k^2}{2M^2}
\]

leads to
\[
\rho_S = \rho_B - \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3k \frac{k^2}{2M^2} = \rho_B \left[1 - \frac{3k_F^2}{10M^2}\right]
\]

where
\[
\rho_B = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d^3k = \frac{\gamma}{6\pi^2} k_F^3.
\]

The scalar density becomes
\[
\rho_S = \rho_B \left[1 - \frac{3k_F^2}{10M^2} + O \left(\frac{k_F^4}{M^4}\right)\right].
\]

Now, we are able to calculate the binding energy per particle
\[
E_B = \frac{\epsilon}{\rho_B} - M
\]
\[
= \frac{g_\omega}{2} V_0 + \frac{g_\sigma}{2} \Phi \frac{\rho_S}{\rho_B} + \frac{3}{4} \sqrt{k_F^2 + M^2} + \frac{1}{4} M^* \frac{\rho_S}{\rho_B} - M
\]
\[
\approx \frac{g_\omega^2}{2m_\omega^2} \rho_B + \frac{g_\sigma^2}{2m_\sigma^2} \rho_B \left[1 - \frac{3k_F^2}{10M^2}\right]^2
\]
\[
+ \frac{3}{4} \left(M - \frac{g_\sigma^2}{m_\sigma^2} \rho_S\right) \left[1 + \frac{k_F^2}{2M^2}\right]
\]
\[
+ \frac{1}{4} \left(M - \frac{g_\sigma^2}{m_\sigma^2} \rho_S\right) \left[1 - \frac{3k_F^2}{10M^2}\right] - M
\]
\[
= \frac{3k_F^2}{10M} + \frac{g_\omega^2}{2m_\omega^2} \rho_B - \frac{g_\sigma^2}{2m_\sigma^2} \rho_B + \frac{g_\sigma^2}{m_\sigma^2} \rho_B \frac{3k_F^2}{10M}
\]

\[
E_B = \frac{3k_F^2}{10M} + \frac{1}{2} \left(\frac{g_\omega^2}{m_\omega^2} - \frac{g_\sigma^2}{m_\sigma^2}\right) \rho_B + \frac{g_\sigma^2}{m_\sigma^2} \rho_B \frac{3k_F^2}{10M}
\]

8.3.3 Determination of the model parameters

In the following we discuss the determination of the model parameters and the physical consequences. In its minimal version of QHD-I the model has only two free parameters, namely the coupling constants \(g_\sigma\) and \(g_\omega\) of the two mesons. The meson masses are constants with \(m_\sigma = 550\text{ MeV}\) and \(m_\omega = 783\text{ MeV}\). The meson-nucleon coupling constants \(g_\sigma\) and \(g_\omega\) are now fixed such that one obtains an optimal description of the saturation behaviour of nuclear matter. Thus the model has to fulfil the following requirements:
CHAPTER 8. MEAN FIELD THEORY

1. A stable ground state of nuclear matter is only possible if the energy density permits a bound state. More precisely, the energy density must be negative for a certain density range: $\frac{\rho_s}{\rho_B} - M < 0$. This condition can be realized, because the model contains two fields. Firstly, the attractive scalar field $\Phi = \frac{g_\sigma}{m_\sigma^2} \rho_S$ and secondly the repulsive vector field $V_0 = \frac{g_\omega}{m_\omega^2} \rho_B$.

2. To ensure a stable ground state, the binding energy $E_B$ must have a minimum. Therefore, we need a large scalar field $\frac{g_\sigma^2}{m_\sigma^2} > \frac{g_\omega^2}{m_\omega^2}$ with $\frac{d}{dk_F} E_B = 0$ at saturation density. This requirement can be fulfilled since the scalar density $\rho_S$, Eq. (8.19), to which the scalar field is directly proportional, saturates with increasing density $\rho_B$ (or increasing $k_F$). The saturation behaviour of the scalar density $\rho_S$ is based on a pure relativistic effect and leads to an additional repulsion at large densities. This we directly see from the $k_F$ expansion of the scalar density where the leading term is proportional to $\rho_B$ but with increasing $k_F$ the higher order negative correction terms come into play. This corrections appear in the same way in the energy density (8.24) and are responsible for the fact that with increasing density the repulsive vector field $V_0$ wins over the attractive scalar field $\Phi$.

3. An adaption of the minimum $\frac{d}{dk_F} E_B = 0$ to empirical data

$$\left(\rho_0 = 0.17 \pm 1\right) \text{ nucleons per fm}^3 \text{ and } E_B = -16 \text{ MeV}$$

yields the final values for the coupling constants (in dimensionless units)

$$C_\sigma^2 = g_\sigma^2 \left(\frac{M^2}{m_\sigma^2}\right) = 267.1, \quad C_\omega^2 = g_\omega^2 \left(\frac{M^2}{m_\omega^2}\right) = 195.9$$

with a nucleon mass of $M = 939 \text{ MeV}$, $(1\text{fm}^{-1} = 197.33 \text{ MeV})$.

The optimal fit leads to $k_F_0 = 1.42 \text{ fm}^{-1}$ for QHD-I. This value is still slightly too large and can be improved by extensions of the QHD-I model.

In this context we want to stress again that the basic fact responsible for the relativistic saturation mechanism is the additional repulsion introduced by the saturation of the scalar density. It is proportional to $\frac{g_\sigma^2}{m_\sigma^2} k_F^3$.

At the saturation point we have a kinetic energy of approximately 20 MeV which means that the model has to satisfy the condition

$$20 \text{ MeV} + \frac{1}{2} \left(\frac{g_\omega^2}{m_\omega^2} - \frac{g_\sigma^2}{m_\sigma^2}\right) \rho_0 = -16 \text{ MeV}$$

This forces the scalar coupling constant $g_\sigma$ to be large. Due to the subtle cancellation effects between attractive scalar and repulsive vector fields also the vector coupling $g_\omega$ must be large. This leads to a feature which is typical for all relativistic nuclear models:
The appearance of large scalar and vector fields with opposite sign is typical for relativistic dynamics. Experimentally the size of these fields is not accessible since only their difference, the optical potential (8.14) and (8.15), can be measured. However, there exists indirect evidences for existence of such large fields, e.g. the large spin-orbit force in finite nuclei. As we will see in Chapter XXXXXXX a large spin-orbit force can very naturally be explained within the relativistic formalism. Just very recently it has been shown (Plohl and Fuchs, 2005 [13]) that the occurrence of these large fields in matter is a direct consequence nucleon-nucleon interaction in free space as soon as the symmetries of the Lorentz group are respected. This fact demonstrated in Fig. 8.5. There the corresponding scalar and vector self-energy components, i.e. fields are calculated in mean field approximation (tree-level) for various modern high precision nucleon-nucleon potentials. Except of QHD-I all these potentials have been adjusted to nucleon-nucleon scattering in free space. For example Bonn A and CD-Bonn are OBE potentials which we discussed in the previous Chapter. In the case of QHD-I the fields are given by $\Sigma_S = \gamma_\sigma \Phi$ and $-\Sigma_0 = \gamma_\omega V_0$. The appearance of large fields in nuclear matter is a direct consequence of the structure of the NN potential, in particular enforced by P-wave scattering, as soon as the symmetries of the Lorentz group are respected.
Remark: extensions of the $\sigma\omega$-model

Figure 8.6: The equations of state of symmetric and pure neutron matter from QHD-I are compared to extended versions of QHD (NL3, DD) and to a microscopic many-body calculation (DBHF).

As already mentioned there exist a variety of extensions of the original $\sigma\omega$-model QHD-I [8] which improve on the saturation properties and the description of finite nuclei. Typically higher order density dependences are introduced by additional terms in the Lagrangian (8.5). By this way new parameters are introduced as well. Three typical examples are:

- **Non-linear (NL) $\sigma\omega$-models** (Boguta, 1982 [9]): an additional non-linear self-interaction term for the $\sigma$ meson is introduced. This leads to scalar terms $\mathcal{L}_{\sigma\omega} + U(\sigma)$ with the scalar potential $U(\sigma) = -\frac{1}{3}B\sigma^3 - \frac{1}{4}C\sigma^4$. By this way the size and the density dependence of the effective mass can be tuned and very successful models for finite nuclei have been constructed [7].

- **Density-dependent (DD) quantum-hadron-dynamics** (Fuchs, Lenske, Wolter, 1995 [11]). Here the scalar and vector coupling-functions in $\mathcal{L}$ are replaced by density dependent vertex functions

$$\frac{g_\omega^2}{m_\omega^2} \rightarrow \Gamma_\omega(\hat{\rho}) \quad \frac{g_\sigma^2}{m_\sigma^2} \rightarrow \Gamma_\sigma(\hat{\rho})$$
where \( \hat{\rho} = \sqrt{\hat{j}_\mu \hat{j}^\mu} \) and \( \hat{j}_\mu = \bar{\psi} \gamma_\mu \psi \). By this way one can efficiently introduce higher order correlations into the Lagrangian.

- Relativistic point coupling models: here the picture of meson exchange is abandoned. A Lagrangian exclusively formulated in terms of nucleon fields and point-like couplings is systematically expanded in powers of nucleon fields and vertices. The vertices are constructed by contractions of the Dirac vertices (scalar, vector, tensor, ...) with derivatives. Now the derivatives instead of the mesons mediate the finite range of the forces.

Figure 8.6 demonstrates for the case of a few selected models the spread of predictions for the nuclear EOS. While the equations of state obtained with QHD-I are relatively unrealistic, two extensions shown here, namely a widely used non-linear version (NL3, [12]) and a density dependent version of QHD (DD, [11]) turned out to be quite successful in the description of finite nuclei properties. This is of course achieved by the price of additional parameters which allow e.g. a better reproduction of the saturation point. Since both models are fine tuned to finite nuclei their EOSs are similar as long as the density is moderate. This is understandable since \( 0 \leq \rho \leq \rho_0 \) is the region where the model parameters are constraint by data. However, what we see as well is that extrapolating to supra-normal densities leads to strong deviations within the phenomenological models. Consequently, the predictive power of these models is very limited when applied to unknown areas. Then one has to rely on approaches where no parameters are fitted to nuclear bulk properties. One example for such a calculation is DBHF which we will meet in the next Chapter. Such type of approaches are ab initio in the sense that the nuclear many-body problem is treated microscopically and the interaction is the bare nucleon-nucleon force as discussed in the previous chapter. The results shown here were obtained with the Bonn A OBE potential [6].

### 8.3.4 Isospin Asymmetric Matter*

Up to now we considered only isospin symmetric matter and pure neutron matter. Both cases are easy to handle since the difference appears only in the degeneracy factor \( \gamma \). However, these are ideal cases. In most nuclei proton and neutron numbers are different and correspondingly, proton and neutron densities are different as well. Even in neutron stars the matter is not ideal neutron matter but always contains a finite fraction of protons.

Thus for a realistic description we are forced to introduce an additional degree of freedom, namely the isospin asymmetry. (For further details see Ref. [14].) The isospin asymmetry is usually characterised by the so-called isospin asymmetry parameter \( \beta \) which is defined as the difference of neutron \( (Y_n) \) and proton \( (Y_p) \) fraction

\[
\beta = \frac{Y_n - Y_p}{\rho_B} = \frac{\rho_n - \rho_p}{\rho_B}.
\]  

(8.25)
The corresponding densities are obtained projecting the baryon density on the isospin quantum numbers $\tau_3 = \pm 1$, i.e.,

$$\rho_p = \bar{\psi} \gamma_0 \frac{1}{2} (1 + \tau_3) \psi, \quad \rho_n = \bar{\psi} \gamma_0 \frac{1}{2} (1 - \tau_3) \psi.$$ 

Thus proton and neutron densities are obtained inserting isospin projection operators. That $\frac{1}{2}(1 \pm \tau_3)$ are indeed projection operators can easily be verified using $\tau_3^2 = 1$. Now we have two different Fermi momenta for protons and neutrons, $k_{F_p}$ and $k_{F_n}$ which are unequal ($k_{F_p} \neq k_{F_n}$) which means that Fermi spheres of the two particle species have different radii. However, the densities are again obtained by integration over the corresponding Fermi spheres:

$$\rho_{p,n} = \frac{2}{(2\pi)^3} \int_0^{k_{F_{p,n}}} d^3k = \frac{2}{6\pi^2} k_{F_{p,n}}^3.$$ 

Note that the sum of proton and neutron densities is the total baryon density $\rho_B = \rho_p + \rho_n$ but this is not true for their Fermi momenta. Now there exists no more a unique Fermi momentum which can be attributed to the total density.

From Eq. (8.25) we recover the two limiting cases of isospin symmetric matter ($\beta = 0$) and pure neutron matter ($\beta = 1$). If the matter is isospin asymmetric with $0 < \beta < 1$ the isospin dependence of the nuclear forces comes into play. In the Weizäcker mass formula (8.4) this dependence is expressed by the empirical asymmetry term which depends on the difference between proton and neutron numbers. The isospin asymmetry introduces now an additional degree of freedom into the energy functional. The means that the energy density $\epsilon$, Eq. (8.17), and the binding energy $E_B$, Eq. (8.24), depend on two parameters, i.e. proton and neutron Fermi momenta, or the total density and the asymmetry parameter $\beta$, respectively.

Next we want to calculate the energy density within Quantum Hadron Dynamics. For this purpose we have to extend QHD-I to include the isospin dependence of the nuclear forces. From Chapter ?? we know two mesons which carry an isospin dependence, namely the pion (pseudoscalar isovector) and the rho-meson (vector isovector). Like in the symmetric case we will disregard the pion since it does not contribute to the mean field energy functional. This is due to the fact that the pion contributes only through exchange or Fock terms which are not taken into account at the mean field level. Thus we are left with the rho-meson.

The $\rho$-meson can simply be added to the Lagrangian given in Eq. (8.5). Since it is a vector meson the coupling looks identical to the $\omega$-meson except the fact that it is of isovector type. This is at least the case as long as we neglect the tensor coupling of the $\rho$ what we actually do. Thus the terms to be added are

$$\mathcal{L}_\rho = -\frac{1}{4} \vec{F}_{\mu\nu} \cdot \vec{F}^\mu\nu_\rho + \frac{1}{2} m_\rho^2 \vec{\rho} \cdot \vec{\rho} - g_\rho \bar{\psi} \gamma_\mu \tau^a \psi \cdot \vec{\rho}^a. \quad (8.26)$$

$\vec{F}_{\mu\nu}$ is the field-strength-tensor for the $\vec{\rho}$ field, defined by Eq. (8.6). However, as an isospin-1 particle the $\rho$ is a three-vector in isospin space and therefore Eq. (8.6) holds separately
for each component. Like the $\omega$ the rho obeys the Proca equation, now with the nucleon isovector current as source term

$$\partial^\nu \tilde{F}_{\mu \nu} + m_\rho^2 \tilde{\rho}_\mu = g_\rho \bar{\psi} \gamma_\mu \tau^3 \psi$$

(8.27)

In the Dirac equation which, like Eq. (8.27), is obtained from the variational principle, the $\rho$-field simply adds to the $\omega$-field

$$\left[ \gamma_\mu (i\partial^\mu - g_\omega \omega^\mu - g_\rho \tilde{\rho}^\mu \cdot \vec{\tau}) - (M - g_\sigma \sigma) \right] \psi = 0 .$$

(8.28)

To continue, we apply again the mean field approximation. To do so, we consider in the following the case where the nucleon field is an eigenstate of $\tau_3$ with eigenvalues $\pm 1$, i.e. the eigenstates are protons and neutrons. This means that only the third component in isospin space of the $\rho$-field survives, $\tilde{\rho}^\mu \rightarrow \rho_3^\mu$. Analogous to the case of the $\omega$ meson, in infinite matter the isovector mean field potential $V_\tau$ is now obtained from Eq. (8.27) setting all gradient terms equal to zero and taking the expectation value of the source term. Hence $V_\tau$ is given by the time-like component of the $\rho$-field

$$V_\tau^\mu = V_0^\tau \delta_{\mu 0} = \frac{g_\rho}{m_\rho^2} \langle \bar{\psi} \gamma_0 \tau_3 \psi \rangle = \frac{g_\rho}{m_\rho^2} (\rho_p - \rho_n) .$$

(8.29)

Thus the Dirac equation reads now

$$\left( \vec{\alpha} \cdot \vec{k} + \beta M^* \right) u(k) = (E - g_\omega V_0 - \tau_3 g_\rho V_0^\tau) u(k)$$

(8.30)

where the energy eigenvalues are now given by

$$E = \pm \sqrt{\vec{k}^2 + M^*^2 + g_\omega V_0 + \tau_3 g_\rho V_0^\tau}$$

(8.31)

From eq. (8.30) we see that the isovector potential acts with different sign on protons and neutrons (due to $\tau_3$ in front).

In the energy density the contributions from protons and neutrons have now to be treated separately:

$$\epsilon(k_F, \beta) = \frac{g_\omega}{2} V_0 \rho_B + \frac{g_\rho}{2} \Phi \rho_S + \frac{g_\rho}{2} V_0^\tau (\rho_p - \rho_n) + \sum_{i=n,p} \left[ \frac{3}{8} E_{F_i}^* \rho_i + \frac{1}{8} M^* \rho_{S_i} \right]$$

(8.32)

with the proton and neutron Fermi energies $E_{F_i}^* = \sqrt{k_{F_i}^2 + M^*^2}$ and the corresponding scalar densities $\rho_{S_i}$. The latter are obtained from Eq. (8.18) with the corresponding Fermi momenta.

In order to separate the isospin dependent part of the energy functional from that of symmetric nuclear matter it is useful to perform a Taylor expansion in terms of $\beta$. In
this case it is more transparent to consider directly the energy per particle \( E_B \). Thus we expand \( E_B \)

\[
E_B(\rho_B, \beta) = E_B|_{\beta=0} + \frac{\partial E_B}{\partial \beta}|_{\beta=0} \beta + \frac{1}{2} \frac{\partial^2 E_B}{\partial \beta^2}|_{\beta=0} \beta^2 + \cdots
\]

\[
= E_B(\rho_B) + E_{\text{sym}} \beta^2 + \mathcal{O}(\beta^4) + \cdots.
\]  

(8.33)

Eq. (8.33) implies that the linear term in \( \beta \) - and also all other odd terms - vanishes. Hence the energy per particle is in leading order given by the contribution from symmetric nuclear matter \( E_B(\rho_B) \) plus the first asymmetric term which is quadratic in \( \beta \). The coefficient of the quadratic term, the so-called *symmetry energy* has in QHD a compact form

\[
E_{\text{sym}} = \frac{1}{6} \frac{k_F^2}{E_F} + \frac{1}{2} \frac{g_p^2}{m_p^2} \rho_B.
\]  

(8.34)

From (8.34) we see that the symmetry energy does not depend on \( \beta \) itself but only on the total density \( \rho_B \). However, this density dependence is crucial, e.g. for the stability of neutron stars, since it determines the behaviour of the equation of state at high baryon densities. This question is one of the hot questions of present research in nuclear structure. We will come back to this problem when we discuss neutron stars in more detail.

In the \( \sigma \omega + \rho \) model the density dependence of \( E_{\text{sym}} \) is simple: a first, trivial term from the kinetic energy and a linear density dependence from the \( \rho \)-meson. Assuming that the higher order terms of order \( \mathcal{O}(\beta^4) \) can be neglected, the symmetry energy given by Eq. (8.34) is just the difference between the nuclear equation of state of neutron matter \( (\beta = 1) \) and that of symmetric nuclear matter \( (\beta = 0) \).

In QHD-I, i.e. without the \( \rho \)-meson, this difference is generated by the larger kinetic energy due the enlarged Fermi momentum of the neutrons. As we can see from Fig. (8.4) already in that case the *additional kinetic energy leads to an EOS which is always unbound*. The isospin dependence of the interaction leads in its simplest form to an additional repulsive term which grows linear with density.

Different density functionals, e.g. extended versions of QHD or microscopic many-body calculations lead in general to a more complex density dependency of \( E_{\text{sym}} \). This fact is demonstrated by Fig. 8.7 where the symmetry energies obtained from the models already shown in Fig. 8.6 are displayed.

---

**Taylor expansion in \( \beta \):**

In order to perform the Taylor expansion we introduce the isovector density \( \rho_3 \)

\[
\rho_3 = \rho_p - \rho_n \quad , \quad \rho_{p/n} = \frac{1}{2}(\rho_B \pm \rho_3)
\]
and go back to the energy density. Making use of

$$\frac{\partial}{\partial \beta} = -\rho_B \frac{\partial}{\partial \rho_3}, \quad \frac{\partial \rho_S}{\partial \rho} = \frac{M^*}{E_F} \quad \text{and} \quad \frac{\partial k_{F_i}}{\partial \rho_i} = \frac{\pi^2}{k_{F_i}^2}$$

one obtains

$$-\frac{\partial E_B}{\partial \beta} = \frac{\partial \epsilon}{\partial \rho_3}$$

$$= \frac{g^2}{2m^2_p} \rho_3 + \sum_{i=n,p} \left[ \frac{3}{8} E_{F_i}^* + \frac{3}{8} k_{F_i} \frac{\pi^2}{E_{F_i}^* k_{F_i}^2} \rho_i + \frac{1}{8} M^{*2} \right] \frac{\partial \rho_i}{\partial \rho_3}$$

$$= \frac{1}{4} \left[ E_{F_p}^* - E_{F_n}^* \right]$$

(8.35)
and

\[
\rho_B \frac{\partial^2 \epsilon}{\partial \rho_3^2} = \frac{(\rho_p + \rho_n)}{8} \left[ \frac{\pi^2}{E^*_F p k_{F_p}} + \frac{\pi^2}{E^*_F n k_{F_n}} \right]
\]

\[
\frac{\partial^2 E_B}{\partial \beta^2} = \frac{1}{8 \cdot 3} \left[ \frac{k_{F_p}^2}{E^*_F p} + \frac{k_{F_n}^2}{E^*_F n} + \frac{k_{F_p} k_{F_n}}{E^*_F p} + \frac{k_{F_p} k_{F_n}}{E^*_F n} \right].
\] (8.36)

Thus we see that for \( \beta = 0 \), i.e. \( \rho_3 = 0 \) and \( k_{F_n} = k_{F_p} \), the first derivative of the binding energy, Eq. (8.35), vanishes and the second one, Eq. (8.36), yields the result of Eq. (8.34). Note that in the present calculation we assumed that the effective mass \( M^* \) does not depend on \( \beta \). This is actually not completely true since there appears an implicit \( \beta \) dependence due to the different values for the corresponding Fermi momenta. However, this \( \beta \) dependence is very small which can also read off from Fig. 8.4 and can therefore be neglected. An explicit \( \beta \) dependence which leads to a splitting of proton and neutron masses arises when a scalar isovector meson is introduced into the theory. This acts like the scalar \( \sigma \)-meson and couples to the effective mass.

Now we can summarise:

- The isovector potential generated by the \( \rho \)-field depends on the difference between proton and neutron densities.

- It acts with different sign on protons and neutrons.

- For \( \rho_n > \rho_p \), which is usually the case in nuclear systems, the isovector mean field is repulsive for neutrons and attractive for protons.
Figure 8.8: Skylab X-ray picture of our sun (source NASA).

8.4 Thermodynamics of nuclear matter

Up to now we considered only systems at zero temperature. Before coming to the formalism for $T > 0$ we should ask the question: what means hot on a nuclear scale? E.g., in its core the sun has a temperature of about $1.5 \cdot 10^7$ Kelvin and at the corona still $1-2 \cdot 10^6$ Kelvin which we would consider as pretty hot. However in nuclear units this corresponds to $10^{-4}$ MeV ($1 \text{ K} = 8.61735 \cdot 10^{-5} \text{ eV}$) which is totally negligible compared to the binding energy in nuclear matter ($|E_B| \sim 16 \text{ MeV}$) or in a finite nucleus ($\sim 8 \text{ MeV}$). Therefore on a nuclear scale the sun can be considered as cold. Even neutron stars which reach temperatures of a few MeV, i.e. $10^5$ times larger than the sun, can be considered as cold to good approximation.

Really hot matter is created in relativistic heavy ion reactions where temperatures from 5 MeV up to about 100 MeV and more can be reached. When the system is heated up in such a way many new degrees of freedom are excited. These are primarily mesons (pions, rho-mesons, kaons,....) but also nucleonic resonances. Therefore one speaks about a hadron gas rather than about nuclear matter. However, QCD implies that even a hadron gas can not be heated up to arbitrarily high temperatures but that there exists a limiting or critical temperature where the hadron gas undergoes a phase transition to a plasma of deconfined quarks and gluons, the so-called Quark-Gluon-Plasma (QGP). Present lattice QCD calculations predict a value of $T_C \sim 170 \text{ MeV}$ for this critical temperature. This value can be reached by present days most energetic heavy ion accelerators. One believes e.g. that at the Relativistic Heavy Ion Collider RHIC at Brookhaven such state of matter has been seen. Since matter created under such conditions is dominated by mesons it is particle-antiparticle symmetric and the net baryon density is practically zero. Thus it reflects the conditions which we believe to have existed in the early universe directly after
Figure 8.9: A sketch of the QCD phase diagram of matter as a function of temperature and baryon density. The $T = 0$ axis corresponds to nuclear matter in nuclei and neutron matter in neutron stars at high densities.

the Big Bang.

To summarise, we have the following temperature scales in nuclear systems:

- sun: $\sim 10^{-4}$ MeV
- neutron stars: few MeV
- supernovae explosions: $\sim 10 - 20$ MeV
- heavy ion collisions: $\sim 10 - 200$ MeV
- quark-gluon plasma: $> 170$ MeV

### 8.4.1 Finite temperature formalism

In the following we will discuss the finite temperature formalism for nuclear (and neutron matter) within the $\sigma\omega$-model (QHD-I). Thermodynamics of large many-body systems can generally be characterised by the grand canonical thermodynamic potential

$$\Omega(\mu, V, T) = -\frac{1}{\beta} \ln Z.$$ 

The partition function $Z$ is given by

$$Z = \text{tr} e^{-\beta(H - \mu B)}$$

where $\mu$ represents the chemical potential and $\beta = \frac{1}{k_B T}$. The QHD-I Hamiltonian has the form
\[ \hat{H} = V \left[ \frac{1}{2} m_\sigma^2 \Phi^2 - \frac{1}{2} m_\omega^2 V_0^2 \right] + g_\omega V_0 \hat{B} + \sum_{\vec{k}, \lambda} \sqrt{\vec{k}^2 + M^*} \left( a_{\vec{k}\lambda}^\dagger a_{\vec{k}\lambda} + b_{\vec{k}\lambda}^\dagger b_{\vec{k}\lambda} \right). \]

It includes the total baryon number \( \hat{B} \) which now has to be calculated by the number of particles minus anti-particles. Therefore we have

\[ \hat{B} = \sum_{\vec{k}\lambda} (a_{\vec{k}\lambda}^\dagger a_{\vec{k}\lambda} - b_{\vec{k}\lambda}^\dagger b_{\vec{k}\lambda}). \]

Figure 8.10: Schematic representation of the Fermi-Dirac distributions at zero temperature and at finite temperature.

Now we have to include the contributions from the antiparticles explicitly into the formalism. At zero temperature the vacuum is given by the Dirac sea completely filled with antiparticles. The vacuum could be subtracted leading to a renormalised density given by particles only. Now the situation is analogous to e.g. \( e^+ e^- \) pair creation discussed in Chapter XXX: due to thermal energy particle-antiparticle pairs are created with a non-vanishing probability. A nucleon with negative energy from the Dirac sea is shifted over the energy gap of \( 2M^* \) and becomes a particle. Together with the remaining hole in the Dirac sea, i.e. the anti-nucleon, this forms a nucleon-anti-nucleon pair. This means that cold vacuum is now replaced by a hot vacuum which contains real particle-antiparticle excitations. The corresponding occupation numbers of particles and anti-particles are given by covariant Fermi-Dirac distributions for particles (see Figure 8.10) and antiparticles

\[ n(\vec{k}, T) = \left[ e^{\beta (E^*(k) - \mu^*)} + 1 \right]^{-1}, \quad \bar{n}(\vec{k}, T) = \left[ e^{\beta (E^*(k) + \mu^*)} + 1 \right]^{-1}. \] (8.37)
The effective chemical potential which enters into Eqs. (8.37) is given by \( \mu^* = \mu - g_\omega V_0 \) and contains also the vector field. At zero temperature the chemical potential is just given by the Fermi energy, i.e. \( \mu^* = E^*_F \), respectively \( \mu = E_F \) with \( E_F \) from (8.16). At finite temperature the chemical potential has to be from Eq. (8.39), i.e. at a given baryon density \( \mu^* \) has to be chosen such that the corresponding integral yields this density.

With the distribution functions we obtain now the energy density in mean field approximation

\[
\epsilon(\rho_B, T) = \frac{1}{2} g_\omega V_0 + \frac{1}{2} g_0 \Phi + \frac{\gamma}{(2\pi)^3} \int d^3k \sqrt{k^2 + M^*} \left[ n(\vec{k}, T) + \bar{n}(\vec{k}, T) \right]
\]

and the vector and scalar baryon densities

\[
\rho_B(T) = \frac{\gamma}{(2\pi)^3} \int d^3k \left[ n(\vec{k}, T) - \bar{n}(\vec{k}, T) \right]
\]

\[
\rho_S(T) = \frac{\gamma}{(2\pi)^3} \int d^3k \frac{M^*}{\sqrt{k^2 + M^*}} \left[ n(\vec{k}, T) + \bar{n}(\vec{k}, T) \right]
\]

The energy density looks almost identical to the zero temperature case, Eq. (8.17). However, now the integral over the kinetic energy has to be carried out over the hot Fermi-Dirac distributions (8.37) which means, that in contrast to (8.20), there exists no analytic solution and the integrals have to be solved numerically.

Figure 8.11 shows the corresponding results for the equation of state at finite temperature, i.e. at temperatures \( T=10 \text{ MeV} \) and \( T=20 \text{ MeV} \). One observes a strong temperature dependence of the EOS in particular at low densities which shifts the minimum upwards. At even higher temperatures the matter becomes unbound.

Generally, the thermodynamic relation which defines the pressure is given by \( dE = -PdV \) which leads to

\[
P = -\frac{\partial E}{\partial V}.
\]

By writing \( \frac{\partial \epsilon}{\partial \rho_B} \) in an expanded form and inserting the above equation, one obtains

\[
\frac{\partial \epsilon}{\partial \rho_B} = \frac{\partial (E/V)}{\partial V} \frac{\partial V}{\partial \rho_B} = \left( -\frac{E}{V^2} + \frac{1}{V} \frac{\partial E}{\partial V} \right) \left( -\frac{V^2}{B} \right) = \frac{\epsilon}{\rho_B} + \frac{P}{\rho_B}.
\]

This leads finally to the relation between energy density and pressure

\[
P = \rho_B \frac{\partial \epsilon}{\partial \rho_B} - \epsilon = \rho_B^2 \frac{\partial}{\partial \rho_B} \left( \frac{\epsilon}{\rho_B} \right).
\]

8.4.2 Thermodynamic consistency

Now we will shortly examine the question of thermodynamic consistency, i.e. if the model or the approximation scheme conserves basic thermodynamic relations. At a first glance this may look trivial but it is in fact a non-trivial question. While the field theoretical
Figure 8.11: Temperature dependent equation of state of isospin symmetric nuclear matter obtained within QHD-I for T=0, 10, 20 MeV.

model Lagrangian (8.5) is certainly consistent with the laws of thermodynamics, it is not a priori clear that this holds as well for the approximations under which this Lagrangian is treated. E.g. the ladder approximation of Brückner theory which will be discussed in Chapter XXX, is not thermodynamically consistent. In the following we will demonstrate that the mean field approximation of QHD is thermodynamically consistent. To do so it is sufficient to consider the $T = 0$ case. At finite temperature the corresponding derivations run completely analogous.

We have learned that the thermodynamic relation $dE = -pdV$ leads to

$$ P = \rho_B \frac{\partial \epsilon}{\partial \rho_B} - \epsilon. $$

On the other hand, from field theory the pressure follows from the energy-momentum-tensor via variation

$$ P = \frac{1}{3} \sum_{i=1}^{3} T_{ii} $$

where

$$ T^{\mu \nu} = -g^{\mu \nu} \mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial (\partial^\mu \Phi_i)} \partial^\nu \Phi_i. $$
Here, $\Phi_i$ represents the corresponding fields $\phi_i = \bar{\Psi}, \Psi, \sigma, \omega$. As a next step we compute the energy-momentum tensor explicitly

$$T^{\mu\nu} = i\bar{\Psi} \gamma^{\mu} \partial^\nu \Psi + \partial^{\mu} \sigma \partial^{\nu} \sigma$$

$$- \frac{1}{2} g^{\mu\nu} (\partial_\lambda \sigma \partial^\lambda \sigma - m_\sigma^2 \sigma)$$

$$+ \partial^{\mu} \omega_\alpha F^{\alpha\nu} - \left[ \frac{1}{4} F_{\lambda\sigma} F^{\lambda\sigma} + \frac{1}{2} m_\omega^2 \omega_\lambda \omega^\lambda \right] g^{\mu\nu}.$$  \hspace{1cm} (8.42)

Considering nuclear matter at rest, $T^{\mu\nu}$ becomes diagonal

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & P & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{pmatrix}.$$  \hspace{1cm} (8.43)

As a next step, energy density and pressure are calculated from (8.42) in mean field approximation which leads to

$$\epsilon = \frac{g_\omega}{2} V_0 \rho_B + \frac{g_\sigma}{2} \Phi \rho_S + \frac{3}{4} E_F^* \rho_B + \frac{1}{4} M^* \rho_S$$  \hspace{1cm} (8.44)

$$P = \frac{1}{3} \sum_i T^{ii} = \frac{1}{3} \left( \frac{\gamma}{(2\pi)^3} \right) \int_0^{\frac{k_F}{\sqrt{k^2 + M^*}}^2} - \frac{g_\omega^2}{2} \Phi \rho_S + \frac{g_\omega^2}{2} V_0 \rho_B$$

$$= - \frac{g_\sigma^2}{2} \Phi \rho_S + \frac{g_\omega^2}{2} V_0 \rho_B + \frac{1}{4} E_F^* \rho_B - \frac{1}{4} M^* \rho_S.$$  \hspace{1cm} (8.45)

Now, we can easily verify that thermodynamic consistency is ensured. That means, that energy density (8.44) and pressure (8.45) satisfy the relation:

$$P = \frac{1}{3} \sum_i T^{ii} = \rho_B \frac{\partial \epsilon}{\partial \rho_B} - \epsilon.$$  

Moreover, with expressions (8.44) and (8.45) we obtain the enthalpy per volume $w$ as

$$w = \epsilon + P = (g_\omega V_0 + E_F^*) \rho_B = E_F \rho_B.$$  \hspace{1cm} (8.46)

**Limiting cases:**

Finally, we discuss some limiting cases:
1. $k_F \gg M^*$ (high density limit): Since the effective mass drops strongly with increasing density this limit can be reached in neutron stars. It is a special realization of the relativistic limit which means that momenta are much larger than the mass. In that case only the highest powers of $k_F$ in (8.44) and (8.45) are taken into account which originate from the term $E^* \rho_B$. This leads to

$$ \epsilon(k_F) \rightarrow \frac{3}{4} \frac{\gamma}{6\pi^2} k_F^4 $$

$$ P = \frac{\epsilon}{3} $$

2. $T \rightarrow 0$ (zero temperature limit):

$$ n(\vec{k}, T) \rightarrow \theta \left( k_F - |\vec{k}| \right) $$

$$ \bar{n}(\vec{k}, T) \rightarrow 0 $$

$$ \mu^* \rightarrow E_F^* $$

3. $T << \mu$ (low temperature limit):

At finite density but small temperatures the Fermi distribution is close to the $\Theta$-function and the temperature dependence gives a correction to the $T = 0$ case

$$ \epsilon(\mu, T) = \epsilon(k_F, T = 0) + \alpha T^2 \rho_B \quad (8.47) $$

where the proportionality factor $\alpha$ contains on the model parameters.

4. $T \rightarrow \infty$ (high temperature limit):

The corresponding equation of state shows the behaviour of a relativistic gas or a black body.

$$ \epsilon = \frac{7\pi^2\gamma}{120} T^4 \propto T^4 $$

$$ P = \frac{\epsilon}{3} $$

$$ \mu^* \rightarrow 0 $$

The relativistic limit for the equation of state of a Fermi gas (e.g. a nucleon or a electron gas) fulfils generally the relation

$$ P = \frac{\epsilon}{3} \quad (8.48) $$

### 8.4.3 Covariance and streaming matter*

In the following we will demonstrate that the Fermi-Dirac distributions (8.37) are indeed covariant, i.e. that they can be written in a manifestly covariant way (see e.g. L. Sehn, Dissertation, 1990).
First we stress that both, the temperature $T$ and the chemical potential $\mu^*$ are Lorentz scalars. The actual values have to be determined in the local rest frame RF of the matter and have the same values in moving frames. It remains to show that the argument $(E^*(k) - \mu^*)$ can be written in a manifestly covariant and physically meaningful way. A covariant generalisation must be able to describe the Fermi-Dirac distributions in moving frames.

Figure 8.12: Boost of a Fermi sphere with radius $k_F$ to the corresponding Fermi ellipsoid in a moving frame which represents uniformly streaming nuclear matter.

For the following discussion we can restrict ourselves to the $T = 0$ case. The generalisation to finite temperature is then straightforward. At $T = 0$ nuclear matter in its rest frame is described by a Fermi sphere $n(\vec{k}) = \Theta(k_F - |\vec{k}|)$. Also here the distribution can more generally be specified by an energy surface via the chemical potential $\mu^*$

$$n(\vec{k}) = \Theta(\mu^* - E^*(\vec{k})) \ .$$

Now we consider a frame MF moving with velocity $+\vec{v}$ with respect to our rest frame. In that frame the matter has an average streaming velocity $+\vec{v}$ and the Fermi distribution is given by a Fermi ellipsoid which will become clear later on. The distribution in MF can isomorph be mapped on the old Fermi sphere in RF by the Lorentz transformation $\Lambda(-\vec{v})$, i.e.

$$E^* = \Lambda^{0\nu}(-\vec{v})k^*_{\nu} = \gamma \left( \sqrt{\vec{k}^*\nu^2 + M^*} - \vec{v} \cdot \vec{k}^* \right) \ .$$

with $k^*_\nu$ given in MF. ($\gamma = 1/\sqrt{1 + \vec{v}^2}$ is the $\gamma$-factor.) Thereby we have to take into account that the spatial components of the vector field are not vanishing in MF but are given by

$$V^\mu = \Lambda(\vec{v})(V_0, \vec{0}) = (\gamma V_0, \vec{v}\gamma V_0) \ .$$

Thus we obtain the canonical momentum distribution

$$E^* = \gamma \left( \sqrt{(\vec{k}^\prime - \gamma\omega\vec{V}^\prime)^2 + M^*} - \vec{v} \cdot (\vec{k}^\prime - \gamma\omega\vec{V}^\prime) \right) \ .$$
To continue we evaluate Eq. (8.50) for the case of matter streaming along the positive $z$-axis in $M$, i.e. $\vec{v} = v \hat{e}_z$. Taking the square of (8.50) one obtains after some simple algebra (making use of $1 - \vec{v}^2 = \gamma^{-2}$):

$$E^{*2} - M^{*2} = |\vec{k}| = k_x^{*2} + k_y^{*2} + \gamma^{-2}(k_z^{*} - v\gamma E^*)^2 .$$  

(8.51)

Introducing the geometrical quantities

$$b = |\vec{k}|, \quad a = \gamma |\vec{k}| \quad \text{and} \quad q = u\gamma E^*$$

we see that eq. (8.51) defines an ellipsoid

$$\frac{k_x^{*2} + k_y^{*2}}{b^2} + \frac{(k_z^{*} - q)^2}{a^2} = 1$$  

(8.52)

with the centre at $\vec{q} = \gamma \vec{v} E^*$ and small and large half axes $b$ and $a$. Eqs. (8.51) and (8.52) define equi-potential surfaces defined by the radius $|\vec{k}|$ in the rest frame which transform to shifted ellipsoids in the moving frame. The surface of the total distribution is given by the maximum momentum, i.e. the Fermi momentum $|\vec{k}| = k_F$ and, correspondingly, $E^*$ is then given by the chemical potential, i.e. the Fermi energy $\mu^* = E^*_F$. The corresponding distributions are shown in Figure 8.12.

Now we have understood that the covariant generalisation of a Fermi sphere is a covariant Fermi ellipsoid which describes uniformly streaming nuclear matter. The momentum distribution of streaming matter is given by a boosted Fermi sphere. For a covariant generalisation of Eqs. (8.49) and (8.50) we introduce the streaming four-velocity

$$u^{\mu} = (\gamma, \gamma \vec{v}) .$$  

(8.53)

The four-velocity is normalised to unity $u^2 = 1$ and in the local rest frame it is just given by $u^{\mu} = (1, \vec{0})$. The covariant form of Eq. (8.49) is then

$$n(\vec{k}) = \Theta(\mu^* - k_\mu^{*} u^{\mu})$$  

(8.54)

The generalisation to finite temperature is now straightforward since just the arguments of the exponential in (8.37) have to be replaced by the covariant expressions:

$$n(\vec{k}, T) = \left[e^{\beta(E^*_\mu u^{\mu} - \mu^*)} + 1\right]^{-1} , \quad \bar{n}(\vec{k}, T) = \left[e^{\beta(k_\mu^{*} u^{\mu} + \mu^*)} + 1\right]^{-1} .$$  

(8.55)

For streaming matter (or in moving frames) the spatial components of the baryon four-vector current do not vanish which leads to the generalisation of Eq. (8.39)

$$j^{\mu}(T) = \frac{\gamma}{(2\pi)^3} \int d^3k \frac{k^{*\mu}}{E^*} \left[n(\vec{k}, T) - \bar{n}(\vec{k}, T)\right] .$$  

(8.56)

In the rest frame (8.56) reduces to (8.39), i.e. $j^{\mu} = (\rho_B^{(RF)}, \vec{0})$. Indeed, the baryon density in the rest frame RF itself is a Lorentz scalar which can be defined as

$$\rho_B^{(RF)} = \sqrt{j_\mu j^{\mu}}$$
which leads to
\[ j^\mu = (\gamma \rho_B^{(RF)}, \bar{\nu} \gamma \rho_B^{(RF)}) = \rho_B^{(RF)} u^\mu. \]

Now we can even generalise the energy-momentum tensor (8.43) to the case of streaming matter. To do so we introduce first the projector orthogonal to the four-velocity \( u^\mu \) (\( \Delta^{\mu\nu} u_\nu = 0 \))
\[ \Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu} \]
which leads to
\[ T^{\mu\nu} = \epsilon g^{\mu\nu} + (\epsilon + P)(u^\mu u^\nu - g^{\mu\nu}) = \epsilon g^{\mu\nu} + \mu \rho_B^{(RF)} \Delta^{\mu\nu} \]
with the chemical potential \( \mu \).

### 8.5 The liquid-gas phase transition

#### 8.5.1 Basic features of phase transitions

A phase transition means usually that a substance behaves qualitatively different under different external conditions. Thermodynamically this information is presented in the form of a phase diagram. Figure 8.13 shows a phase diagram for \( \text{H}_2\text{O} \) (not to scale). Figure is taken from [16].

![Phase diagram for H₂O](image_url)
of a *phase diagram* in which the external conditions are calibrated in terms of so-called *control parameters*.

The most familiar example is water, i.e. $\text{H}_2\text{O}$, which occurs in the three phases of ice, water and steam. In that case the control parameters are temperature $T$ and pressure $P$ and the different phases occupy different regions in the phase diagram shown in Fig. 8.5.1. The lines mark the various coexistence curves $P(T)$ where two phases are in equilibrium. A phase transition such as melting or boiling is observed when moving along a path in the $(T, P)$ plane which intersects such curves.

There exist two special points in that diagram, namely the *triple point* $(T_{Tr} = 273.16 \text{K}, P_{Tr} = 600 \text{Nm}^{-2})$ where all three phases coexist, and the *critical point* $(T_C = 647 \text{K}, P_c = 2.21 \times 10^7 \text{Nm}^{-2})$, where the separation of liquid from vapour disappears and the two fluid phases become indistinguishable.

Generally we can distinguish between three different types of phase transitions:

- **1st order phase transition:**
  This is the general case for a one-component system as long as $T < T_C$. It is e.g. the transition between liquid and vapour. A first order transition is characterised by a non-vanishing latent heat and interface tension. The classification follows because entropy and volume are both first derivatives of the Gibbs free energy $G(T, P)$

\[
S = -\left.\frac{\partial G}{\partial T}\right|_P ; \quad V = \left.\frac{\partial G}{\partial P}\right|_T \quad (8.58)
\]

and show discontinuities $\Delta S > 0$ and $\Delta V > 0$ in a 1st order transition.

- **2nd order phase transition:**
  At the critical point the transition becomes second order, which means that singularities occur in the specific heat $C_P$ and the isothermal compressibility $\kappa_T$. Both are related to second derivatives of the free energy

\[
C_P = -\left.\frac{T \frac{\partial^2 G}{\partial T^2}}{P}\right|_P ; \quad \kappa_T = -\left.\frac{1}{V} \frac{\partial^2 G}{\partial P^2}\right|_T . \quad (8.59)
\]

At the critical point these quantities diverge.

- **Cross over:**
  At temperatures above the critical temperature $T_C$ the two phases, a dilute liquid or a dense gas are thermodynamically indistinguishable. They occupy the same volume, i.e. $\Delta V = 0$ and $\frac{\partial P}{\partial \rho_B} > 0$. 

Remark: Thermodynamic compressibility

It should be noticed that the isothermal compressibility $\kappa_T$ defined in Eq. (8.59) is different from the definition of the compression modulus $K$, Eqs. (8.22) and (8.23), used in nuclear physics. Inserting the Gibbs free energy ($V = N/\rho_B$)

$$G = E - TS + PV$$

we can rewrite $\kappa_T$ as

$$\kappa_T = -\frac{1}{V} \left. \frac{\partial V}{\partial P} \right|_T = \rho_B \left. \frac{\partial}{\partial \rho_B} \left( \frac{1}{\rho_B} \right) \right|_T = \frac{1}{\rho_B} \left. \frac{\partial \rho_B}{\partial P} \right|_T$$

which leads to the relation

$$K = 9 (\rho_B \kappa_T)^{-1}.$$  

Therefore the compression modulus $K$ is sometimes denoted as incompressibility.

The conditions for a phase coexistence can easily be derived from the Gibbs-Duhem relation which holds in both phases, i.e.

$$0 = SdT - VdP + Nd\mu \quad \text{or} \quad 0 = sdT - dP + \rho d\mu.$$  

For two phases in equilibrium the entropy has to be maximal. Since the entropy is an extensive quantity we can simply add the contributions of both subsystems

$$0 = dS^{(1)} + dS^{(2)} = \sum_{i=1,2} \left[ \frac{1}{T^{(i)}} dE^{(i)} + \frac{P^{(i)}}{T^{(i)}} dV^{(i)} - \frac{\mu^{(i)}}{T^{(i)}} dN^{(i)} \right]$$  

From (8.60) one obtains after some simple algebra the Gibbs conditions for two phases in thermodynamical equilibrium

$$T^{(1)} = T^{(2)}, \quad P^{(1)} = P^{(2)}, \quad \mu^{(1)} = \mu^{(2)}$$  

which means that the intensive parameters temperature, pressure and chemical potentials have to be equal.

As already mentioned, a typical feature of a first order phase transition is the occurrence of latent heat during the coexistence phase. In the so-called caloric curve which shows the temperature dependence of a substance as a function of the excitation energy per particle the latent heat is reflected by a plateau. Figure 8.5.3 shows the caloric curve of water at constant pressure.

At this point it should be stressed that all features of phase transitions discussed up to now are valid in infinite systems. Water or macroscopic systems in general, can safely be considered as infinite. Nuclei are, however, microscopic systems which consist at maximal of a few hundred particles. In microscopic systems the clear features of a phase transition are
washed out to some extent by finite size effects. Hence a clear first order phase transition in an infinite system may be reduced to a smooth cross over in a finite system. The importance of finite size effects can be read of from Fig. 8.5.3 where in addition to the caloric curve of 'infinite' water that of a bulk of only 20 molecules is shown as well. When such a finite bulk of molecules melts, i.e. undergoes a phase transition from a solid to a liquid, the temperature plateau is much less pronounced though still visible and the transition temperature is shifted to below. The latter is due to the fact that the finite cluster can more easily evaporate single molecules from the surface.

There exist a striking analogy between the nuclear phase diagram and that of water. The reason behind is due to the fact that the forces are of similar shape although they are of totally different strength and act on a totally different length scale. However, the intra-molecular forces in water are of a so-called van der Waals type which is similar to the nucleon-nucleon interaction. Van der Waals forces contain a long-range part, an intermediate range attractive and a short range repulsive part. The corresponding equation
Figure 8.15: Isotherms (in arbitrary units) of a van der Waals gas, e.g. water, are compared to those of a nucleon gas in the $P - V$ diagram. The nuclear interaction is of so-called Skyrme type. The figure is taken from [18].

The equation of state reads

$$P \left(1 + \frac{a}{V^2}\right)(V - b) = NT$$

where the parameters $a$ describes the intra-molecular attractive force while $b$ accounts for the excluded volume due to the finite molecular size (equivalent to the nucleon hard core). The correspondence of states between the molecular and the nuclear systems is demonstrated in Fig. 8.5.1 where the corresponding equation of state of a van der Waals gas are compared to those of nuclear matter (derived from non-relativistic so-called Skyrme forces).

The analogy is also reflected in the fact that a nucleus behaves to good approximation like a liquid drop. As expressed by the Weizäcker mass formula the binding energy has a bulk part and a part due to the surface tension. If the system is heated up it undergoes a phase transition from a liquid to vapour, i.e. to a gas of nucleons. At high temperatures the kinetic energy becomes large compared to the nuclear forces and the system behaves more and more like a non-interacting Fermi gas.

What makes this nuclear liquid-gas phase transition in particular interesting is the fact that both phases, cold nuclear Fermi liquids on the one hand and a nuclear gas consisting of free
nucleons and a few light clusters on the other hand, exist in nature and are experimentally accessible, as we will see below.

### 8.5.2 Spinodal instabilities

![Figure 8.16: Isotherms of the equation of state of nuclear matter in the $P - \rho$ phase diagram. The spinodal region is indicated by dashed curves. The figure is taken from [17].](image)

To continue we consider the finite temperature phase diagram of nuclear matter shown in Fig. 8.5.2. Here the equation of state is given in terms of pressure, (8.41) and (8.45), instead of energy density. In thermodynamics that is the more familiar way to present the EOS, although both representations are essentially equivalent. The Figure shows the results from an already historic calculation (Sauer, Chandra, Mosel, 1976 [17]), of course not based on QHD but on simpler non-relativistic nuclear forces. However, the qualitative picture is still the same.

In order to understand the phase diagram qualitatively, we consider first the $T = 0$ case. At very low densities the system is very dilute and the nucleons interact only weakly. Hence
they move almost freely and the pressure is given by that of a free Fermi gas
\[ P_{FG}(k_F, T = 0) = \frac{1}{4} \rho_B \sqrt{k_F^2 + M^2} - \frac{1}{4} M \rho_S \simeq \frac{\gamma}{48\pi^2 M} k_F^5. \]

It is always positive
\[ P_{FG}(k_F) > 0. \]

The fact that the pressure can become negative is exclusively due to the nuclear interactions. At the saturation point, i.e. the minimum of the energy per particle (see Fig. 8.4), the pressure has to vanish
\[ P(\rho_0) = 0. \]

Below that point the pressure is negative which means that the attractive forces are superior and the matter want to shrink, i.e. to compress itself. Above that point the repulsion takes over and the matter wants to expand.

Besides this, we see than in a region between \( 0.01 < \rho_B < 0.08 \) the matter behaves somewhat strange since the derivative of the pressure with respect to density is negative which means that the pressure decreases when the matter expands. The region with
\[ \frac{\partial P}{\partial \rho_B} < 0 \]
is called the unstable or spinodal region. It is unstable in the sense that the matter does not want to be homogeneous but to clump and to break into pieces. It is the region of phase coexistence where droplets of different size coexist with a gas of single nucleons.

To determine the isotherms shown in Fig. 8.5.2 within the framework of QHD, one has to calculate the pressure according to Eqs. (8.41) and (8.45)
\[ P = -\frac{g_5^2}{2} \Phi_{PS} + \frac{g_2^2}{2} V_0 \rho_B + \frac{1}{3} \frac{\gamma}{(2\pi)^3} \int \frac{d^3 k \sqrt{k^2 + M^2}}{\sqrt{k^2 + M^2}} \left[ n(\vec{k}, T) + \bar{n}(\vec{k}, T) \right]. \]  

This can only be done numerically and leads to qualitatively similar results than shown in Fig. 8.5.2.

From the thermodynamic relation
\[ TdS = PdV + dE - \mu dN \]
we obtain also the entropy density \( s = S/V \)
\[ s = (\epsilon + P - \mu \rho_B) \]

### 8.5.3 The nuclear caloric curve

In this Subsection we discuss experimental evidence for the nuclear liquid-gas phase transition. Since the cold nucleus is a sort of a liquid it has to be heated up. Such an excited
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Figure 8.17: Schematic view of the temporal evolution of a heavy ion reaction at intermediate energies where a liquid-gas phase transition occurs.

hot bulk of nucleons can be created in energetic heavy ion collisions.

The caloric curve of nuclear matter has been measured in a famous experiment performed at the Gesellschaft für Schwerionenforschung (GSI) in Darmstadt/Germany using the ALADIN spectrometer [?]. If one tries to derive the caloric curve - as shown in Fig. 8.5.3 for water - for a nuclear system one has to face two major problems connected to the two axes of this plot:

- How to control the excitation energy per nucleon?
- What is a suitable thermometer?

Both are difficult problems but the first is the more easy one.

The temporal evolution of a heavy ion reaction is sketched in Fig. 8.5.3. The size of projectile and target and the impact parameter of the reaction define the so-called participant region in which the nucleons are strongly interacting and most of the kinetic beam energy is deposited. The remaining part, the so-called spectator matter suffers also energy transfer but the reaction is much less violent.

The participant region is significantly heated up and undergoes the transition to a gas of nucleons. Afterwards the matter in the participant region expands and breaks into fragments. In the phase diagram this means that the matter cools down and enters the spinodal region of coexistence of liquid (intermediate and heavy mass fragments) and gas (single nucleons and light fragments such as He).
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In contrast to the participant region the heating of the spectators proceeds in a more adiabatic way. Also the excited spectator matter breaks finally into fragment. The excitation energy can be deduced measuring the complete fragmentation pattern including the kinetic energy of the fragments. An important point is thereby that the invariant features of the spectator multi-fragment decay are consistent with equilibration of the excited systems prior to their decay. This justifies interpretations in statistical or thermodynamical terms.

Less straightforward is the determination of a nuclear temperature. Nuclei are closed systems without an external heat bath. Consequently, the temperature of the system cannot be pre-determined but has to be reconstructed from observables. For a microcanonical ensemble, the thermodynamic temperature of a system may be defined in terms of the total-energy state density. An experimental determination of the state density and its energy dependence is, however, impossible. Therefore, nuclear temperature determinations take recourse to ‘simple’ observables which constitute reasonable approximations to the true thermodynamic temperature.

Figure 8.18: Caloric curve of nuclei determined by the dependence of the isotope temperature $T_{HeLi}$ on the excitation energy per nucleon. The figure is taken from [19].
In the ALADIN experiment the temperature was extracted via the double yield ratio of two isotope pairs, \( \frac{Y_1}{Y_2} \) and \( \frac{Y_3}{Y_4} \), differing by the same number of neutrons

\[
R = \frac{Y_1/Y_2}{Y_3/Y_4} = a \cdot e^{(B_1-B_2)-(B_3-B_4)/T}.
\]

Here, \( B_i \) denotes the binding energy of particle species \( i \) and the constant \( a \) contains known spins and mass numbers of the fragments. A large sensitivity of this thermometer can be achieved if the constant \( b = (B_1 - B_2) - (B_3 - B_4) \) is larger than the typical temperature to be measured. Due to the strong binding energy of \( \alpha \) particles, particularly large values for \( b \) are obtained if a \( ^3\text{He}/^4\text{He} \) ratio is involved. For the second yield the ratio \( ^6\text{Li}/^7\text{Li} \) has been chosen. Thus the finally extracted temperature has been dubbed as \( T_{\text{HeLi}} \).

Figure 8.5.3 shows the isotope temperature as a function of the total excitation energy per nucleon. The measured caloric curve can be divided into three distinctly different sections:

- **Nuclear liquid:**
  Up to an excitation energy of about 3 MeV the rise of \( T_{\text{HeLi}} \) is compatible with the low-temperature approximation of a fermionic system. According to Eq. (8.47) the excitation energy per particle \( E_x/A = (E(T) - E(0))/A \) is proportional to \( T^2 \):

  \[
  \langle E_x \rangle / \langle A \rangle \simeq \frac{1}{\alpha} T^2
  \]

  where the constant \( \alpha \) has empirical been determined from excited compound nuclei as \( \alpha \sim 8 \div 13 \text{ MeV} \).

- **Coexistence phase:**
  From 3 MeV to 10 MeV \( T_{\text{HeLi}} \) shows an almost constant value of about 4.5-5 MeV. This plateau indicates the occurrence of latent heat.

- **Gas phase:**
  Beyond a total excitation energy of 10 MeV per nucleon, a steady rise of \( T_{\text{HeLi}} \) with increasing \( \langle E_x \rangle / \langle A \rangle \) is seen. The rise follows roughly that of a classical nucleon gas

  \[
  \langle E_x \rangle / \langle A \rangle = \frac{3}{2} T.
  \]

  The excitation energy has thereby to be corrected for the binding energy per fragment which is about 6 MeV and for quantum effects due to Fermi statistics.
Bibliography