From Elasticity to Hypoplasticity: Dynamics of Granular Solids

Yimin Jiang$^{1, 2}$ and Mario Liu$^1$

$^1$Theoretische Physik, Universität Tübingen, 72076 Tübingen, Germany
$^2$Central South University, Changsha 410083, China

(Received 22 January 2007; published 5 September 2007)

“Granular elasticity,” useful for calculating static stress distributions in granular media, is generalized by including the effects of slowly moving, deformed grains. The result is a hydrodynamic theory for granular solids that agrees well with models from soil mechanics.

DOI: 10.1103/PhysRevLett.99.105501 PACS numbers: 81.40.Lm

Granular media has different phases that, in dependence of the grain’s ratio of deformation to kinetic energy, may loosely be referred to as gaseous, liquid, and solid. The first phase is relatively well understood: Moving fast and being free most of the time, the grains in the gaseous phase have much kinetic, but next to none elastic, energy [1]. In the denser liquid phase, say in chute flows, there is less kinetic energy, more deformation, and a rich rheology that has been scrutinized recently [2].

In statics, with the grains deformed but stationary, the energy is all elastic. This state is legitimately referred to as solid because static shear stresses are sustained. If granular solid is slowly sheared, the predominant part of the energy remains elastic. There is no theory capable of accounting for its statics and dynamics, and no understanding helps to render its physics transparent.

Two grains in contact are initially very compliant, being so little material is being deformed. As this geometric fact should also hold on larger scales, for many grains, diverging compliance at diminishing compression is a basic characteristic of granular solids, and the reason it is sensible to abandon the approximation of infinitely rigid grains. Starting from this observation, a theory termed GE (for “granular elasticity”) was constructed to account for static granular stress distributions. Taking the energy $w$ as a function of $u_{ij}$, the elastic contribution to the total strain field $e_{ij}$, we specify [3]

$$w = \mathcal{B}\sqrt{\Delta (2\Delta^2 / 5 + u_{ij}^2 / \xi)},$$

(1)

with $\Delta = -u_{\ell \ell}$, $u_{ij}^2 = u_{ij}^0 u_{ij}^0$, $u_{ij}^0 = u_{ij} - \frac{1}{2} u_{\ell \ell} \delta_{ij}$ (The notations: $a_{ij}^0 = a_{ij} - \frac{1}{2} a_{ij} \delta_{ij}$ and $a_{ij}^2 = a_{ij}^0 a_{ij}^0$, with any $a_{ij}$ is employed throughout this Letter). The elastic coefficient $\mathcal{B}$, a measure of overall rigidity, is a function of the density. Denoting $\rho_\ell$ as the granular material’s bulk density, and $e = \rho_\ell / \rho - 1$ as the void ratio, we take $\mathcal{B} = \mathcal{B}_0 (2.17 - e)^2 / [1.3736(1 + e)]$, with $\mathcal{B}_0$, $\xi > 0$ two material constants. The elastic energy $w$ contributes $\pi_{ij} = -\partial w / \partial u_{ij}$ to the total stress $\sigma_{ij}$. And since the elastic stress is the only contribution in statics, force balance reads $\nabla_j \sigma_{ij} = \rho G_i$. This was solved for three classical cases: silos, sand piles, and granular sheets under a point load, resulting in rather satisfactory agreement to experiments, see [4]. Moreover, the energy $w$ (with $P = \frac{1}{3} \pi_{\ell \ell}$) is convex only for $\pi_{\ell \ell} / P \leq \sqrt{2 / \xi}$, implying no elastic solution is stable beyond it. Identifying this as the yield surface gives $\xi = 5/3$ for natural sand.

When granular solid is being slowly sheared, we must expect a qualitative change of its behavior: In addition to moving with the large-scaled velocity $v_i$, the grains also move and slip in deviation of it—implying a small but finite granular temperature $T_g$. As a result, some of the grains are temporarily unjammed, with enough time to decrease their deformation. This depletes the elastic energy and relaxes the static stress. Stress relaxation is typical of viscoelastic systems such as polymers. Granular media are similar, but they possess a relaxation rate that vanishes with $T_g$. This is the reason they return to perfect elasticity when stationary. The basic physics of granular solids, viscoelasticity at finite $T_g$, is in fact epitomized by a sand pile, which holds its shape when unperturbed, but fails to do so when tapped. A set of differential equations termed granular solid hydrodynamics (GSH) is derived consistently below starting from GE, with this simple physics as the only additional input.

Conservation of density and momentum always holds,

$$\partial_t \rho + \nabla_i (\rho v_i) = 0, \quad \partial_t (\rho v_i) + \nabla_j (\sigma_{ij} + \rho v_i v_j) = \rho G_i,$$

(2)

where $G_i$ is the gravitational constant. In granular gas or liquid, the stress $\sigma_{ij}$ has the same structure as in the Navier-Stokes equation, though the viscosity is a function of the shear. In granular solid, the stress is not usually taken to be given in a closed form. Instead, constitutive relations are employed. These relate the temporal derivatives of stress and strain, giving $\partial_t \sigma_{ij}$ as a function of $v_{ij} = \frac{1}{2} \times (\nabla_i v_j + \nabla_j v_i)$ and density (where $\partial_t$ is often replaced by an objective derivative, say, from Jaumann).

Hypoplasticity, or HPM (for hypoplastic model), is a modern, well-verified, yet comparatively simple theory of soil mechanics [5]. It is quite realistic in the above specified regime of solid dynamics, though less appropriate for determining static stress distributions. The starting point is the rate-independent constitutive relation,
\[ \partial_t \sigma_{ij} = H_{ijk \ell} u_{k \ell} + \Lambda_{ij} v_{\ell}^0 v_{\ell}^0 + \epsilon (v_{\ell}^0)^2, \]  
\text{(3)}

where the coefficients \( H_{ijk \ell}, \Lambda_{ij}, \epsilon \) are functions of \( \sigma_{ij}, \rho \), specified using experimental data mainly from triaxial apparatus. Great efforts are invested in finding accurate expressions for them, of which a recent set \[5\] is \( \epsilon = 1/3, \)
\[ H_{ijk \ell} = f (F^2 \delta_{ik} \delta_{j \ell} + a^2 \sigma_{ij} \sigma_{\ell \ell} / \sigma_{nn}^2). \]  
\text{(4)}

below for their differences. Calculating \( @t \sigma_{ij} \), \( @t v_{\ell} \), accounting for the relaxation of the elastic strain \( u_{ij} \), is rather more consequential. Equation \( (9) \) is obtained by taking the derivative of Eq. \( (1) \), \[ \pi_{ij} \equiv -\partial w / \partial u_{ij} = \sqrt{\Delta (B \Delta - 2 A u_{ij}^0) + A (u_{ij}^0 / 2 \Delta) \delta_{ij}}, \]  
with the relaxation times given as \( 1/\tau = 2 B A \sqrt{\Delta}, 1/\tau_1 = 3 B_1 \sqrt{\Delta (B + \frac{1}{3} A u_{ij}^0 / \Delta^2)}. \) (Note \( A = B / \epsilon \).) The Onsager coefficient \( \alpha \) is, for simplicity, taken as a scalar.

The transport coefficients \( \eta, \eta_\nu, \zeta, \zeta_\nu, \tau, \tau_1, \alpha \) are functions of thermodynamic variables: \( \rho, T_g, u_{ij}. \) We assume they are strain independent, while noting three points: (i) Constant \( \tau, \tau_1 \) implies strain-dependent \( B, B_1 \). Choosing the former as constant, and not the latter, is at this stage, before more experimental data are considered, preliminary, and not crucial: All four figures below retain their form also with \( B, B_1 \) taken as constant. (ii) As \( X_{ij} \) vanishes identically with \( T_g \), an obvious and simple assumption is
\[ 1/\tau = \lambda T_g, 1/\tau_1 = \lambda_1 T_g, \]  
\text{(10)}

with \( \lambda, \lambda_1, \tau_1/\tau = \lambda/\lambda_1 \) functions of the density, but independent from strain and \( T_g. \) (iii) Being reactive, \( \alpha \) is not restricted in its magnitude by that of \( \beta, \beta_1 \). It may stay constant while \( 1/\tau, 1/\tau_1 \) vary—though it must eventually vanish for \( 1/\tau, 1/\tau_1 \to 0 \), as \( \alpha = 0 \) in statics.

The above hydrodynamic theory is closed if we amend it with an equation of motion for \( T_g \). In thermodynamics, the energy change \( dw \) from all microscopic, implicit variables is subsumed as \( T ds \), with \( s \) the entropy and \( T \equiv \partial w / \partial s \) its conjugate variable. From this, we divide out the kinetic energy of granular random motion, executed by the grains in deviation from the ordered, large-scale motion, and denote it as \( T_g ds_g \), calling \( s_g, T_g \equiv \partial w / \partial s_g \) granular entropy and temperature. In other words, we consider two heat reservoirs, \( s_g \) and \( s \), with the second accounting for the rest of all microscopic degrees of freedom, especially phonons. In equilibrium, \( s_g \) is part of the total entropy, and (because of its comparatively small number of degrees of freedom) essentially zero. But when the granular system is being tapped or sheared, rendering \( T_g \gg T \) as a result, \( s_g \) turns into a leaky, intermediate heat reservoir, fed by macroscopic motion while leaking energy into \( s \)—an important reason for the novelty of granular physics. Note \( s_g \) is the generalization of the entropy in granular gas \[1\] to solid densities. The balance equations for \( s, s_g \) are \( \partial_t s + \nabla_k (s u_k) = R / T, \partial_t s_g + \nabla_k (s_g u_k) = R_g / T_g, \) where
\[ R = \eta v_{\ell}^0 + \zeta v_{\ell}^0 + \beta \pi_{\ell}^0 + \pi_{1 \ell} + \gamma T_g^2, \]  
\text{(11)}
\[ R_g = \eta s_g v_{\ell}^0 + \zeta s_g v_{\ell}^0 - \gamma T_g^2. \]  
\text{(12)}

(Diffusion of \( T, T_g \) is neglected, but easily included when needed.) The first two terms in either entropy productions \( R, R_g \) account for viscous heating, how any shear and compressional flows fill up both heat reservoirs at the same time. The next two terms in \( R \) account for the
dissipation from stress relaxation, as given by transient elasticity. The term $\gamma T_g^2$ (with $\gamma > 0$) describes how the energy seeps from $s_g$ into $\dot{s}$, say, because the kinetic energy of random motion is converted into phonons. The form $\gamma T_g^2$ is determined by it being positive definite, or alternatively, by taking $s_g$ as a slow variable, and requiring it to satisfy the relaxation equation $\dot{s}_g = -\gamma T_g^2$.

With Eqs. (1), (2), (6), (7), and (9)–(12), GSH is complete. It especially contains the equilibrium case $\sigma_{ij} = \pi_{ij}$ in which the dissipative fields vanish, $\alpha_{ij}^D, X_{ij} = 0$. Off equilibrium, these two fields are finite, and we calculate $\partial_i \sigma_{ij} = (1 - \alpha) \partial_i \pi_{ij} = (1 - \alpha) M_{ijk\ell} \partial_k u_{\ell k}$

$$= (1 - \alpha) M_{ijk\ell} \left[ (1 - \alpha) v_{\ell k} - u_{\ell k}^0 / \tau - \delta_{k\ell} u_{\ell k} / \tau_1 \right].$$

(13)

As mentioned above, the energy $w$ loses its convexity at $\pi_s / P = \sqrt{6/5}$, and no static, elastic solution is possible beyond this ratio. Therefore, it was identified as yield. Given Eq. (13), the same identification holds dynamically: The loss of convexity implies that one of the six eigenvalues of $M_{ijk\ell} \equiv -\delta_{ij} / \partial_k \partial_{\ell k}$ (written as a 6 x 6 matrix) vanishes at this point, and a strain rate along the associated direction yields vanishing stress rate.

For $R_g = 0$, when $s_g$ is being produced and leaking at the same rate, we have a stationary $T_g$ given as

$$T_g = \sqrt{\eta_g / \gamma} \sqrt{u_s^2 + (\xi_1 / \eta_g) u_\ell^2}.$$ (14)

Inserting Eqs. (10) and (14) into (13), we retrieve Eq. (3), with

$$H_{ijk\ell} = (1 - \alpha)^2 M_{ijk\ell}, \quad \epsilon = \xi_1 / \eta_g.$$ (15)

$$\Lambda_{ij} = (1 - \alpha) M_{ijk\ell} \left[ (\tau / \tau_1) \Delta \delta_{k\ell} - u_{\ell k}^0 / \lambda_3 \sqrt{\eta_g / \gamma} \right] \Lambda_{ij}^\Lambda.$$ (16)

HPM has 43 free parameters ($36 + 6 + 1$ for $H_{ijk\ell}, \Lambda_{ij}, \epsilon$), all functions of the stress and density. Expressed as here, the stress and density dependence are essentially determined by $M_{ijk\ell}$ that (with $\xi = 5/3$ and $B_0 = 8500$ MPa) is a known quantity [4]. For the four free constants, we take

$$1 - \alpha = 0.22, \quad \frac{\tau}{\tau_1} = 0.09,$$

$$\frac{\xi_1}{\eta_g} = 0.33, \quad \lambda_3 \sqrt{\eta_g / \gamma} = 114,$$

(17)

to be realistic choices, as these numbers yield satisfactory agreement with HPM. Their significance is as follows: $\xi_1 / \eta_g = 0.33$ implies shear flows are 3 times as effective in creating $T_g$ as compressional flows. $\tau / \tau_1 = 0.09$ means, plausibly, that the relaxation rate of shear stress is 10 times higher than that of pressure. For a purely elastic system, Eq. (3) is replaced by $\partial_i \sigma_{ij} = M_{ijk\ell} \partial_k u_{\ell k}$. Therefore, the factor $(1 - \alpha)^2$ accounts for an overall, dynamic softening of the static compliance tensor $M_{ijk\ell}$, a known effect in soil mechanics [8]. Finally, $\Lambda$ controls the stress relaxation rate for given $T_g$, and $\sqrt{\eta_g / \gamma}$ how well shear flow excites $T_g$.

Together, $\lambda_3 \sqrt{\eta_g / \gamma} = 114$ determines the relative weight of plastic versus reactive response. (Note $|\Lambda_{ij}| / |H_{ijk\ell}| \sim |u_{\ell k}^0| \times 114/(1 - \alpha)$ is, for $|u_{\ell k}^0|$ around $10^{-3}$, of order unity).

Next, we compare Eqs. (15) and (16) to (4) and (5) in their results with respect to “response envelopes,” a standard test in soil mechanics for rating constitutive relations [5]. Axial symmetry of the triaxial geometry is assumed, with $\sigma_{ij}, v_{ij}$ diagonal, and $\sigma_1 \equiv \sigma_{xx} = \sigma_{yy}, \sigma_2 \equiv \sigma_{zz}, \sigma_3 \equiv \sigma_{yy}, v_1 \equiv v_{xx} = v_{yy}, v_2 \equiv v_{zz}, v_3 \equiv v_{zz}$, $P \equiv \frac{1}{3} \sigma_1 + \frac{1}{3} \sigma_3, \sigma = \sigma_3 - \sigma_1, \sigma_3' \equiv 2 q^2, \delta = -(2 v_1 + v_3) dt$. Starting from a point in the stress space (spanned by $\sigma_1, \sigma_3$ in Fig. 1 and $\sigma_3, P$ in Fig. 2), one deforms the system for a constant time $dt$, at given strain or stress rates, while recording the change in the conjugate quantity. Varying the direction, the input is a circle around the starting point, but the response envelopes show deformation characteristic of the system, or the constitutive relation.

FIG. 2. The change in strain $d\gamma, d\sigma$ for a given stress rate starting from different points in the stress space, spanned by $\sigma_1, P$. The amplitude of the stress rate $\sqrt{d\sigma^2 + d\sigma^2}$ is constant. See Fig. 3 for an explanation of the “flow direction.”
to be rated. Figures 1 and 2 show, respectively, the responding stress and strain envelopes, for the void ratio \( e = 0.66 \), calculated using GSH and HPM. The similarity in stress dependence and anisotropy is obvious.

In Fig. 3, one strain envelope is blown up for a more detailed comparison, using the extended version of response envelope as given in [9]. Here, the applied stress rate is reversed at halftime, such that the system returns to the starting point in stress space at the end. The responding strain change, depicted as deflected, straight dotted lines, does not return to the origin. Both GSH and HPM predict that the end points from all angles of stress changes (some of the angles are given at the deflection points) form a straight line \( OA \). (Instead of a line, a narrow ellipse is reported in the 2D simulation of [9]. This may be a result of the fact that the stationarity of \( T_g \) is briefly violated when the stress rate is reversed, during which the system is rather less plastic.) \( OA \)'s angle \( \sigma \) in strain space is usually referred to as the “flow direction,” while the direction in stress space, along which the plastic deformation is largest (with the strain starting at \( O \) and ending at \( A \)) is called the “yield direction” \( \phi \). Since they are not equal, the flow rule is “nonassociated.” In Fig. 4, the flow direction \( \sigma \), the yield direction \( \phi \), and the maximal plastic strain (the length of \( OA \)), are displayed as functions of \( \sigma_i/P \), with \( P = 0.2 \) MPa. Again, the similarity between both theories is obvious.

We take all this to be a preliminary confirmation for the basic idea of slowly sheared granular solids being viscoelastic, and also for GSH as the appropriate hydrodynamic theory. Next, it should be interesting to use GSH for circumstances in which \( T_g \) is not stationary and the stress rate possesses a more complicated form than that given by Eqs. (3), (15), and (16). These include especially sudden changes in the direction of the strain rate [8], such as in cyclic loading or sound propagation. Also, one needs to understand whether GSH holds at transitions from granular solid to liquid, from \( \nu_{ij} = 0 \) to \( \nu_{ij} \neq 0 \) for a stationary stress, \( \partial_t \nu_{ij} = 0 \), in phenomena such as shear banding.

![Diagram of yield direction, flow direction, and the maximal plastic strain (length of OA), versus \( \sigma_i/P \), for \( P = 0.2 \) MPa, calculated employing GSH and HPM, respectively.](image)

FIG. 4. Yield direction, flow direction, and the maximal plastic strain (length of OA), versus \( \sigma_i/P \), for \( P = 0.2 \) MPa, calculated employing GSH and HPM, respectively.

[6] H. Temmen, H. Pleiner, M. Liu, and H. R. Brand, Phys. Rev. Lett. 84, 3228 (2000); H. Pleiner, M. Liu, and H. R. Brand, Rheol. Acta 43, 502 (2004). (Nonlinear convective terms such as \( u_{ik}v_{jk} \) or \( u_{ik}\pi_{jk} \) are not displayed, because granular media typically consist of hard grains, with \( u_{ik} \ll 1 \). So these terms are negligible when compared to \( u_{jk} \) and \( \pi_{jk} \). The total strain \( \varepsilon_{ij} \), of course, is usually quite large.)
[8] A. Niemunis and I. Herle, Mech. Cohes.-Fric. Mater 2, 279 (1997). It is perhaps useful to note that a softening effect may also be achieved by a \( T_g \) dependence of the elastic coefficient \( B \), with \( \xi = 5/3 \) unchanged.